A sequence of bounded linear operators $A_n \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is said to converge to an operator $A \in \mathcal{H}$: uniformly if $A_n \to A$ with respect to the operator norm on $\mathcal{B}(\mathcal{H})$; strongly if $A_n x \to Ax$ strongly in $\mathcal{H}$ for every $x \in \mathcal{H}$.

(a) Give an example of a sequence of operators that converges strongly but not uniformly.

**Proof.** Remember we did this in 201A. Solution due to Eric!

- Let $T_n = T^n$ where $T$ is the left shift operator on $\ell^2(\mathbb{Z})$ given by
  $$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

First, we will show that $T_n$ converges strongly to 0. Then we will show that it doesn’t converge uniformly to zero.

Pick any sequence $x = (x_1, x_2, \ldots)$ in $\ell^2(\mathbb{Z})$; we have that $x_n \to 0$ as $n \to \infty$.

Remember that the $\ell^2(\mathbb{Z})$-norm of $T_n x$ is given as follows:

$$||T_n x|| = \left( \sum_{i=n}^{\infty} x_n^2 \right)^{1/2} < \infty,$$

which monotonically approaches 0 as $n \to \infty$.

So, $T_n$ converges strongly to the zero operator.

- If $T_n$ converged uniformly, it would have to agree with the strong limit we have found above i.e., it suppose to be 0. We can calculate the norm of $||T_n||$ by first noting that clearly $||T_n|| \leq 1$. To prove that indeed the norm of $T_n$ is 1, we do the usual trick: if we take any sequence $s_n \in \ell^2(\mathbb{T})$ that begins with $n$ zeros then $||T_n s_n|| = ||s_n||$, which implies that $||T_n|| \geq 1$. Thus, $||T_n|| = 1$ for all $n \in \mathbb{N}$.

Therefore we can conclude that, $T_n$ does not converge uniformly since we cannot have

$$||T_n|| = 1 \to 0.$$

\[ \square \]

(b) Give an example of a sequence of operators that converges weakly but not strongly.
Proof. Remember we did this in 201A. Solution due to Eric!

• Let $S_n = S^n$ where $S$ is the right shift operator on $\ell^2(\mathbb{Z})$ given by
  \[ S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots). \]

Our proof here will mirror the structure of the argument in part (a). First, we show that $S_n$ converges weakly to zero. Secondly, we will show that it cannot converge strongly to zero, implying that it must not converge strongly at all.

• Given any bounded linear functional $\phi$ on $\ell^2(Z)$ and any $x \in \ell^2(Z)$, we can linearly decompose the action of $\phi$ on $x$ as the action of the components $\phi_i$ of $\phi$ on the components $x_i$ of $x$ given by $\phi_i(x_j) = a_i x_j$ and find
  \[
  \phi(S_n x) = \phi(0, \ldots, 0, x_1, x_2, \ldots)
  = \sum_{i=n+1}^{\infty} \phi_i(x_{i-n})
  = \sum_{i=n+1}^{\infty} a_i x_{i-n}
  \leq \left( \sum_{j=1}^{\infty} x_j^2 \right)^{(1/2)} \left( \sum_{i=n+1}^{\infty} a_i^2 \right)^{(1/2)},
  \]
  which goes to zero since $a_i \to 0$ as $n \to \infty$.

Thus, $S_n$ converges weakly to 0. However, it is clear from the definition that $||S_n x|| = ||x||$ for all $n \in \mathbb{N}$. Therefore, we cannot have

\[ S_n x \to 0 \text{ as } n \to \infty. \]

Hence, $S_n$ converges weakly to 0, but not strongly.

\[ \square \]

2. A subset $E$ of a vector space $X$ is said to be convex if
  \[ \lambda x + (1 - \lambda) y \in E \quad \forall x, y \in E, 0 \leq \lambda \leq 1. \]

(a) Show that a strongly closed, convex subset of a Hilbert space is weakly closed.

Proof. If $x_n \rightharpoonup x$, where $\{x_n\}_n \subset E$, by Mazur’s theorem, there is a sequence of $\{y_n\}_n \subset E$ of finite convex combination of $\{x_n\}_n$ such that $y_n \to x$. Note that the sequence $\{y_n\}_n$ is really a subset of $E$ because we were given that $E$ is convex! So $x \in E$, because $E$ is strongly closed. Therefore, we conclude that $E$ is weakly closed.

\[ \square \]

(b) Show that every strongly closed, convex subset of a Hilbert space contains a point of minimum norm.
Proof. Suppose $E$ is a closed convex set in a Hilbert space $\mathcal{H}$. Let

$$d = \inf_{x \in E} \|x\|.$$ 

- If $d = 0$, then we can find a sequence $\{x_n\}_n$ so that
  $$\lim_n \|x_n\| = 0.$$ 

Then we find $\{x_n\}_n$ is convergent to 0. Therefore 0 is a limit point of $E$. Since $E$ is closed we find $0 \in E$. If $\|x\| = 0$, then $x = 0$. Hence 0 is the unique minimum point when $d = 0$.

- Suppose $d > 0$. Then we can find a sequence $\{x_n\}_n$ so that $\lim_n \|x_n\| = d$. By the parallelogram’s law, we find

\begin{equation}
\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - 4\left\|\frac{x_n + x_m}{2}\right\|^2.
\end{equation}

Since $E$ is convex, $x_n \in E$ for all $n \in \mathbb{N}$, then we find $(x_n + x_m)/2 \in E$. By the definition of $d$, we see that

\begin{equation}
d^2 \leq \left\|\frac{x_n + x_m}{2}\right\|^2.
\end{equation}

(0.1) and (0.2) imply

$$\|x_n - x_m\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$ 

Since $\lim_{n \to \infty} \|x_n\| = d$, (or $\lim_{n \to \infty} \|x_n\|^2 = d^2$), then for each $\epsilon > 0$, we can find $N_\epsilon > 0$ so that $n \geq N_\epsilon$,

$$0 \leq \|x_n\|^2 - d^2 < \frac{\epsilon^2}{4},$$

which implies that whenever $n, m \geq N_\epsilon$, we have

$$\|x_n - x_m\|^2 < \left(2d^2 + \frac{\epsilon^2}{2}\right) + \left(2d^2 + \frac{\epsilon^2}{2}\right) - 4d^2 = \epsilon^2.$$ 

We proved that $\{x_n\}_n$ is a Cauchy sequence.

- Since $\mathcal{H}$ is a Hilbert space, then we can find $x \in \mathcal{H}$ so that
  $$\lim_{n \to \infty} x_n = x.$$ 

We find $x$ is a limit point of $E$. Since $E$ is closed, then $x \in E$. We also have

$$\lim_{n \to \infty} \|x_n\| = \|x\| = d,$$

and this is given by the continuity of the norm. Hence $x$ is indeed a minimum point.
• Suppose $y$ is another point with $\|y\| = d$. Again, using the parallelogram’s law, we find

$$0 \leq \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x + y}{2}\right\|^2$$

$$= 4d^2 - 4\left\|\frac{x + y}{2}\right\|^2$$

$$\leq 4d^2 - 4d^2 = 0.$$ 

We find $\|x - y\| = 0$, which implies that $x = y$. We conclude that the minimum point is unique. \qed

And alternative proof is to observe that a strongly closed, convex set is weakly closed, and the norm is weakly lower semi-continuous and coercive, so it attains its minimum on any weakly closed set.