1. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator, show that 

$$\|A^n\| = \|A\|^n$$

for every $n \in \mathbb{N}$. (You can use the results proved in class.)

Proof.

- Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint linear operator. Then we can prove that $A^k$ is also a bounded self-adjoint operator for any $k \in \mathbb{N}$. Why? The answer is because of the following argument.

Assume that $\|A\| = M$, where $M$ is some constant greater than zero. Then by induction we can conclude the following:

$$\|A^n\| = \|A^{n-1}A\| \leq \|A^{n-1}\| \|A\| \leq \|A\|^n = M^n$$

and

$$(A^n)^* = (A^{n-1}A)^* = A^*(A^{n-1})^* = A(A^{n-1}) = A^n.$$ 

Hence, we got that indeed $A^n$ is also a bounded, self-adjoint linear operator. The linearity of $A^k$ follows right away from the linearity of $A$.

- The spectral radius of a bounded self-adjoint operator is given by its norm; more precisely, remember that

$$\|A^k\| = r(A^k) = \lim_{n \to \infty} \|A^{kn}\|^{1/n}.$$ 

Relabelling the indices so that $u = nk$, we have that

$$\|A^k\| = \lim_{u \to \infty} \|A^u\|^{k/u} = \left( \lim_{u \to \infty} \|A^u\|^{1/u} \right)^k = r(A)^k = \|A\|^k.$$ 

Hence, we are done proving our problem. \qed
2. Define \( m : (0, 1) \to (0, 1) \) be
\[
m(x) = \begin{cases} 
0 & \text{if } 0 < x < 1/4 \\
2(x - 1/4) & \text{if } 1/4 \leq x \leq 3/4 \\
1 & \text{if } 3/4 < x < 1
\end{cases}
\]
and let \( M : L^2(0, 1) \to L^2(0, 1) \) be the multiplication operator \( Mf = mf \).

Determine the spectrum of \( M \) and classify it into point, continuous and residual spectrum. Describe the eigenspace of any eigenvalues in the point spectrum.

\[\text{Proof. due to Amanda.}\]

- First let's establish some properties about the operator \( M \). Clearly, \( M \) is a self-adjoint operator since we have the following:
\[
\langle f, Mg \rangle = \int_0^1 f(x)m(x)g(x)dx = \int_0^1 m(x)f(x)g(x)dx = \langle Mf, g \rangle.
\]
Note that here the fact that \( m(x) \) is a real valued function, i.e., \( \overline{m(x)} = m(x) \), helped us to conclude that \( M \) is self-adjoint.

- \( M \) is self-adjoint, implies that \( \sigma_r(M) = \emptyset \) and \( \sigma(M) \subseteq [-\|M\|, \|M\|] \).
This results that I have just stated can be found in Prop.9.8 and Lemma 9.13 form your textbook.

- Looking at the definition of \( m(x) \), we can see that the maximum value that \( m \) can attain is 1 and we can also observe that \( 0 \leq m^2(x) \leq 1 \) for all \( x \in [0, 1] \).
Hence, applying the definition of the norm of \( M \) we get:
\[
\|M\| = \sup_{\|f\|=1} \|Mf(x)\| = \sup_{\|f\|=1} \left( \int_0^1 m^2(x)|f(x)|^2dx \right)^{1/2} \leq \sup_{\|f\|=1} \left( \int_0^1 |f(x)|^2dx \right)^{1/2} = 1
\]

- If we take any \( f(x) \in L^2(0, 1) \) with support on \( (3/4, 1) \) such that \( \|f(x)\| = 1 \), we see that \( \|Mf(x)\| = 1 \), so \( \|M\| = 1 \). Hence, \( \sigma(M) \subseteq [-1, 1] \). Notice that for any nonzero function \( f(x) \) with support on \( (0, 1/4) \) that \( Mf(x) = 0 \), so these are functions with eigenvalue 0.

Furthermore, we know such functions exist. Similarly, any nonzero function \( f(x) \) with support on \( (3/4, 1) \) satisfies \( Mf(x) = f(x) \) so these are functions
with eigenvalue 1. So \{0, 1\} \subseteq \sigma_p(M).

• Consider any \(\lambda \in [-1,0)\). Then since \(m(x) \geq 0\) for all \(x\), it follows that \(m(x) - \lambda \geq -\lambda > 0\) for all \(x\). Since \((M - \lambda I)f(x) = (m(x) - \lambda)f(x)\), if \((M - \lambda I)f(x) = 0\) then \(f(x) = 0\) a.e., so it follows that \(M - \lambda I\) is one-to-one.

Also, if \(g(x) \in L^2(0,1)\) then \(\frac{1}{m(x) - \lambda}g(x)\) is well-defined and in \(L^2(0,1)\), so
\[
(M - \lambda I)\frac{1}{m(x) - \lambda}g(x) = g(x).
\]
So \(M - \lambda I\) is also onto.

Hence, for \(\lambda \in [-1,0)\) we see that \(\lambda \in \rho(M)\), so \(\sigma(M) \subseteq [0,1]\). If \(\lambda \in (0,1)\), then we still have \(M - \lambda I\) is one-to-one since \(m(x) - \lambda\) is nonzero for all but one value of \(x\) so \((M - \lambda I)f(x) = (m(x) - \lambda)f(x) = 0\) implies \(f(x) = 0\).

However, \(M - \lambda I\) is not onto. If it were onto, then \(1 \in \text{ran}(M - \lambda I)\). Then there would be an \(f(x) \in L^2(0,1)\) such that
\[
(M - \lambda I)f(x) = (m(x) - \lambda)f(x) = 1 \quad \text{so} \quad f(x) = \frac{1}{m(x) - \lambda}
\]
However, there is some \(x_0 \in (1/4, 3/4)\) such that \(m(x_0) = 2(x_0 - 1/4) = \lambda\).

It follows that
\[
\int_{1/4}^{3/4} \frac{1}{(2(x - 1/4) - \lambda)^2}dx \to \infty.
\]
Since
\[
\|f(x)\|_2^2 = \int_0^{1/4} \frac{1}{\lambda^2}dx + \int_{1/4}^{3/4} \frac{1}{(2(x - 1/4) - \lambda)^2}dx + \int_{3/4}^{1} \frac{1}{(1 - \lambda)^2}dx
\]
we see that \(f(x) \notin L^2(0,1)\). Hence, \(M - \lambda I\) is not onto.

Therefore, \(\lambda \in \sigma(M)\) for \(\lambda \in (0,1)\). Particularly, since \(M - \lambda I\) is one-to-one, we know that \(\lambda \in \sigma_v(M)\) or \(\lambda \in \sigma_r(M)\). From class we know that \(\sigma_r(M) = \emptyset\) since \(M\) is self-adjoint. Hence, it must be that \(\lambda \in \sigma_c(M)\).

Therefore, the spectrum \(\sigma(M) = [0,1]\) where \(\sigma_p(M) = \{0, 1\}\) and \(\sigma_c(M) = (0,1)\). \(\square\)
3 Suppose that \( \{\lambda_n\} \) is a sequence of complex numbers such that \( \lambda_n \to 0 \) as \( n \to \infty \) and define the operator \( K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) by

\[
K(x_1, x_2, \ldots, x_n, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots)
\]

(a) Prove that \( K \) is a compact operator. (Recall that a set is precompact iff it is totally bounded)

*Proof.* Let \( \{\lambda_n \to 0\} \) and let \( K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) be defined by

\[
K(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots).
\]

We want to show that \( K \) is a compact operator. We know that finite-rank operators are compact operators (you have proved this as a homework problem in 201A) and we also know that the uniform limit of compact operators is a compact operator.

Let

\[
K_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})
\]

be given by

\[
K_n(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, 0, 0, \ldots).
\]

Then \( K_n \) is finite-rank operator and hence, a compact operator. Note that

\[
\|K - K_n\| = \sup_{\|x\|=1} \|(K - K_n)x\|
\]

\[
= \sup_{\|x\|=1} \|(0, \ldots, 0, \lambda_{n+1} x_{n+1}, \lambda_{n+2} x_{n+2}, \ldots)\|
\]

\[
= \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2
\]

\[
\leq \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i|^2 |x_i|^2
\]

\[
\leq \sup_{\|x\|=1} \sup_{j=n+1} |\lambda_j|^2 \sum_{i=n+1}^{\infty} |x_i|^2
\]

\[
\leq \sup_{\|x\|=1} \sup_{j=n+1} |\lambda_j|^2 \|x\|^2
\]

\[
\leq \sup_{j=n+1} |\lambda_j|^2,
\]

which can be made arbitrarily small for sufficiently large \( n \), i.e., \( \to 0 \) as \( n \to \infty \).

Therefore we can conclude that \( K_n \to K \) uniformly and \( K \) is compact. \( \square \)
(b) Let $P_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the orthogonal projection onto the $n$th component,

$$P_n(x_1, x_2, \ldots, x_n, \ldots) = (0, 0, \ldots, 0, x_n, 0, \ldots)$$

In what sense (uniformly, strongly, weakly) does the sum $\sum_{n \in \mathbb{N}} \lambda_n P_n$ converge to $K$? Does your answer change in $\lambda \not\to 0$ as $n \to \infty$?

**Proof.** Since $P_n(x_1, x_2, \ldots) = (0, \ldots, 0, x_n, 0, \ldots)$, using the same $K_n$ as the one we defined above, we have

$$K_n = \sum_{k=1}^{n} \lambda_n P_n.$$ 

We have probed above that $K_n \to K$ uniformly.

• Now we discuss the case when $\lambda_n$ does not converge to 0, but it is still bounded.

quire In this case we can see that $K_n$ doesn’t converge uniformly to $K$, and the reason is the following:

$$\lim_{n \to \infty} \|K - K_n\| = \lim_{n \to \infty} \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2 \leq \lim_{n \to \infty} \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i|^2 |x_i|^2,$$

and since we assumed that $\lambda_n \not\to 0$ then there exists an $\epsilon > 0$ such that for every $M \in \mathbb{N}$ we can find $m > M$ such that $|\lambda_m|^2 > \epsilon$. Picking $x = e_m$, we get that

$$\lim_{n \to \infty} \|K - K_n\| > \epsilon.$$ 

This proves that indeed $K_n$ doesn’t converge uniformly to $K$.

quire In the same context, meaning the case when $\lambda_n$ does not converge to 0, but it is still bounded, we want to see if $K_n$ converges strongly to $K$. For this, let’s fix $x \in \ell^2(\mathbb{N})$. Assume also that $|\lambda_i| \leq G$ for every $i \in \mathbb{N}$, where $G$ is a positive constant.

Therefore

$$\lim_{n \to \infty} \| (K - K_n) x \| = \lim_{n \to \infty} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2 \leq \lim_{n \to \infty} G^2 \sum_{i=n+1}^{\infty} |x_i|^2.$$ 


But, this last term converges to 0 as \( n \to \infty \), since \( x \in \ell^2(\mathbb{N}) \), by definition implies
\[
\sum_{i=n+1}^{\infty} |x_i|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 \to 0 \text{ as } n \to \infty.
\]
Hence, \( K_n \to K \) strongly. This clearly implies that \( K_n \to K \) weakly.

- If we assume that the sequence \( \{\lambda_n\}_{n} \) is not bounded either, then from the work we did above, we can see that \( K_n \not\to K \) weakly. Implicitly this tells you that \( K_n \) doesn’t converge strongly or uniformly to \( K \).

\[\Box\]

4. Determine the spectra of the left and right shift operators on \( \ell^2(\mathbb{N}) \)

\[
S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots),
\]

\[
T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),
\]

and classify them into point, continuous, or residual spectrum.

**Proof. due to Amanda.**

We have previously shown that \( \|S\| = 1 = \|T\| \). So by a theorem if \( \lambda \in \mathbb{C} \) such that \( |\lambda| > 1 \), then \( \lambda \in \rho(S) \) and \( \lambda \in \rho(T) \). Now I will prove the following four claims:

(a) **Claim:** \( S - \lambda I \) is one-to-one for all \( \lambda \in \mathbb{C} \) such that \( |\lambda| \leq 1 \).

**Proof.** I will treat this in two cases. If \( \lambda = 0 \) then \( Sx = 0 \) implies \( x_i = 0 \) for all \( i \) so \( x = 0 \). Hence, \( S - \lambda I \) is one-to-one. Suppose that \( 0 < |\lambda| \leq 1 \). Then \( (S - \lambda I)x = 0 \), implies
\[
(0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots)
\]

Since \( \lambda \neq 0 \), \( 0 = \lambda x_1 \) implies \( x_1 = 0 \). Since \( \lambda x_n = x_{n-1} \), a simple induction shows that \( x_n = 0 \) for all \( n \). Hence, \( x = 0 \), so \( S - \lambda I \) is one-to-one for all \( \lambda \) such that \( |\lambda| \leq 1 \).

\[\Box\]

(b) **Claim:** \( S - \lambda I \) is not onto for \( |\lambda| \leq 1 \).

**Proof.** Note that if \( \lambda = 0 \) that \( e_1 \) is clearly not in ran\((S - \lambda I) = \) ran\((S)\). Suppose that \( \lambda \neq 0 \). Then \( (S - \lambda I)x = e_1 \) implies that \( -\lambda x_1 = 1 \) and \( x_{n-1} - \lambda x_n = 0 \) for \( n \geq 2 \). An induction argument shows that \( x_1 = -1/\lambda \), and \( x_n = -1/\lambda^n \). However,
\[
\|x\| = \sum_{n=1}^{\infty} \left( \frac{1}{|\lambda|^2} \right)^n
\]
Note the above sum is a geometric series with $r = \frac{1}{|\lambda|}$. Since $0 < |\lambda|^2 \leq 1$, it follows that $\frac{1}{|\lambda|^2} \geq 1$. Hence the above sum diverges, so $x \notin \ell^2(\mathbb{N})$. Therefore, $e_1 \notin \text{ran}(S - \lambda I)$ so this is not onto. 

\(\square\)

(c) **Claim:** $T - \lambda I$ is one-to-one for $|\lambda| = 1$.

**Proof.** Suppose that it was not one-to-one. Then there would be some nonzero $x \in \ell^2(\mathbb{N})$ such that $(T - \lambda I)x = 0$ or

$$(x_2, x_3, x_4, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots)$$

Hence, $x_2 = \lambda x_1$ and a simple induction shows that $x_n = \lambda^{n-1}x_1$. Therefore, $x \neq 0$ implies $x_1 \neq 0$, and since $|\lambda| = 1$ we see that

$$\|x\| = \sum_{n=0}^{\infty} |x_1|^2|\lambda|^{2n} = \sum_{n=0}^{\infty} |x_1|^2 \rightarrow \infty$$

a contradiction so it must be that $x = 0$, and it follows that $T - \lambda I$ is one-to-one. \(\square\)

(d) **Claim:** $T - \lambda I$ is not one-to-one for $|\lambda| < 1$.

**Proof.** By a similar argument as above, we see that any nonzero $x$ that satisfies $(T - \lambda I)x = 0$ is of the form

$$x = (x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \ldots)$$

Choose $x_1 = 1$. Then

$$\|x\| = \sum_{n=0}^{\infty} (|\lambda|^2)^n$$

Since the above is a geometric series with $r = |\lambda|^2 < 1$, we see that the sum converges, so $x \in \ell^2(\mathbb{N})$ is nonzero and satisfies that $(T - \lambda I)x = 0$. Hence, $T - \lambda I$ is not one-to-one. \(\square\)

Note that claims (a) and (b) show that $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Recall that $\text{ran}(S - \lambda I)$ is dense iff $\ker((S - \lambda I)^*) = \ker(T - \overline{\lambda}I) = \{0\}$. Since $|\overline{\lambda}| = |\lambda|$, claims (c) and (d) show that $\text{ran}(S - \lambda I)$ is dense iff $|\lambda| = 1$. Therefore, $\sigma_c(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_r(S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Also, from the above work we get $\sigma_p(S) = \emptyset$.

Note that claim (d) shows us that $\sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Since $\sigma(T)$ is a closed set, we know that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ (as from above we know that $|\lambda| > 1$ implies $\lambda \in \rho(T)$). For $|\lambda| = 1$ we know that $\ker((T - \lambda I)^*) = \ker(S - \overline{\lambda}I) = \{0\}$ by claim (a). Hence, we know that $\text{ran}(T - \lambda I)$ is dense. Therefore, it must be that $\sigma_c(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_r(T) = \emptyset$. \(\square\)