201B, Winter '11, Professor John Hunter
Homework 1 Solutions

Please note that there are some remarks regarding this homework posted on Professor's Hunter webpage!

1. If $1 \leq p < q < \infty$, show that $L^p(\mathbb{T}) \supset L^q(\mathbb{T})$. Give an example of a function in $L^p(\mathbb{T}) \setminus L^q(\mathbb{T})$.

Proof. **a)** Assume that $1 \leq p < q < \infty$. Then considering $1 \leq r \leq \infty$ such that $\dfrac{1}{r} + \dfrac{1}{q} = \frac{1}{p}$, we have $\dfrac{p}{r} + \dfrac{p}{q} = 1$ i.e., $\dfrac{1}{\frac{r}{p}} + \dfrac{1}{\frac{q}{p}} = 1$, which implies that $\dfrac{r}{p}$ and $\dfrac{q}{p}$ are Holder conjugates.

Applying **Holder's inequality** we get:

$$\begin{aligned}
\|f\|_p^p &= \int_{\mathbb{T}} |f|^p \, dx \\
&= \int_{\mathbb{T}} |f|^p \times 1 \, dx \\
&\leq \left(\int_{\mathbb{T}} (|f|^p)^{\frac{q}{p}} \, dx \right)^{\frac{p}{q}} \times \left(\int_{\mathbb{T}} 1 \, dx \right)^{\frac{p}{r}} \\
&\leq \left[\left(\int_{\mathbb{T}} (|f|)^q \, dx \right)^{\frac{1}{q}} \right]^p \times \left(\int_{\mathbb{T}} 1 \, dx \right)^{\frac{p}{r}} \\
&= \|f\|_q^p \times \mu(\mathbb{T})^{\frac{p}{r}} \\
&= \|f\|_q^p \times \mu(\mathbb{T})^{\frac{q}{q-p}}
\end{aligned}$$

Hence, we proved that for $1 \leq p < q < \infty, \quad L^q(\mathbb{T}) \subset L^p(\mathbb{T})$.

$\square$

Proof. **b)** Next we are asked to give an example of a function in $L^p(\mathbb{T}) \setminus L^q(\mathbb{T})$ i.e., the inclusion $L^q(\mathbb{T}) \subset L^p(\mathbb{T})$ is proper. One easy example is given by:

$$f(x) = \frac{1}{x^{1/q}}.$$

Note that f is measurable almost everywhere. Therefore, f is Lebesque measurable. Now it remains to prove that indeed $f \in L^p$, and is not in L^q. Therefore,

$$\|f\|_p = \left(\int_0^{2\pi} \frac{1}{x^{p/q}} \right)^{1/p}$$

Since $q > p$ it follows that $0 < \frac{p}{q} < 1$. Hence $1 - \frac{p}{q} > 0$ and

$$\|f\|_p = \left(\frac{2\pi^{1-p/q}}{1-p/q}\right)^{1/p} < \infty$$

Therefore $f \in L^p$. But,

$$\|f\|_q = \left(\int_0^{2\pi} \frac{1}{x}\right)^{1/q} = \lim_{b\to 0}\left(\int_b^{2\pi} \frac{1}{x}\right)^{1/q}$$

which diverges as b approaches 0. Therefore $f \notin L^q$. Hence $f \in L^p(\mathbb{T}) \setminus L^q(\mathbb{T})$.

$\square$

2. **b)** If $f, g \in L^2(\mathbb{T})$, show that $\|f * g\|_\infty \leq \|f\|_2 \|g\|_2$ and deduce that
$$f * g \in C(\mathbb{T}).$$

Proof. **b)** First we observe that from part **a)** we have that $f * g$ is measurable. Now, if f, and $g \in C(\mathbb{T})$ then our result follows right away by applying **Holder's inequality**, we get the desired result. More explicitly,

$$|\int_{\mathbb{T}} f(x-y)g(y)\,dy| \leq \int_{\mathbb{T}} |f(x-y)||g(y)|\,dy \leq \|f\|_{L^2(\mathbb{T})}\|g\|_{L^2(\mathbb{T})}$$

Taking the supremum of the inequality we just got, we get:

$$\|f * g\|_\infty \leq \|f\|_{L^2(\mathbb{T})}\|g\|_{L^2(\mathbb{T})}$$

But the problem is when $f, g \in L^2(\mathbb{T})$. Here you remember that there exist sequence of continuous functions $\{f_n\}_n$ and $\{g_n\}_n$ that approximate f and g in the sense that $\|f - f_n\|_2 \to 0$ as $n \to \infty$, and respectively $\|g - g_n\|_2 \to 0$. For each n the convolution function $f_n * g_n$ is a continuous function since for each n f_n and g_n are continous functions.
Note that we have the following inequality that holds:

$$\begin{aligned}\|f_m * g_m - f_n * g_n\|_\infty &\leq \|(f_m - f_n)g_n\|_\infty + \|f_n(g_n - g_m)\|_\infty \\ &\leq \|f_m - f_n\|_2\|g_n\|_2 + \|f_n\|_2\|g_n - g_m\|_2 \\ &\leq M(\|f_m - f_n\|_2 + \|g_n - g_m\|_2).\end{aligned}$$

Note that here we used that $\|g_n\|_2 \leq M$ and $\|f_n\|_2 \leq M$ for some constant $M > 0$ because we know that the sequences $\{f_n\}_n$ and $\{g_n\}_n$ converge in $L^2(\mathbb{T})$. Hence, we proved above that the sequence $\{f_n * g_n\}_n$ is a Cauchy sequence in $C(\mathbb{T})$ and since this is a complete metric space, it follows that $\{f_n * g_n\}_n$ converges uniformly to a continuous map $f * g$ that must belong to $C(\mathbb{T})$.

$\square$

a) If $f, g \in L^1(\mathbb{T})$, show that $f * g \in L^1(\mathbb{T})$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proof. **a)** We know f and g are in $L^1(\mathbb{T})$ and we want first to prove that that the convolution $f * g$ is a measurable function and that this function is in $L^1(\mathbb{T})$.
Let $F_1(x, y) = f(x - y)$ and $F_2(x, y) = g(y)$, then F_1 and F_2 are both measurable and hence $F(x, y) = F_1(x, y)F_2(x, y)$ is measurable.
Then by **Tonelli theorem**, we know

$$\int_{\mathbb{T}\times\mathbb{T}} |F(x, y)|\, dx\, dy = \int_{\mathbb{T}} |f| dx \int_{\mathbb{T}} |g|\, dy = \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})} < \infty.$$

We find $F \in L^1(\mathbb{T}\times\mathbb{T})$. Then by **Fubini theorem**, the function $F(x)$ defined by

$$F(x) = \int_{\mathbb{T}} F(x, y) dy = \int_{\mathbb{T}} f(x - y)g(y) dy$$

is measurable a.e. and

$$\|F\|_{L^1(\mathbb{T})} \leq \int_{\mathbb{T}} \int_{\mathbb{T}} |F(x, y)|\, dy\, dx = \|f\|_{L^1(\mathbb{T})} \|g\|_{L^1(\mathbb{T})} < \infty.$$

Since $F(x) = (f * g)(x)$, thus we prove our assertion.

Note: that $(L^1(\mathbb{T}), *)$ is indeed a Banach algebra.

$\square$

Remark. Here we used the following facts (the results are given on $\mathbb{R}^n$, but as I mentioned above, they still work on $\mathbb{T}$):

(a) If $f(x)$ is measurable on $\mathbb{R}^n$, then the function $F(x, y) = f(x - y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.
Let F_1 be the function defined by $F_1(x, y) = f(x)$, then for each $a \in \mathbb{R}$,

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : F_1(x, y) > a\} = \{x \in \mathbb{R}^n : f(x) > a\} \times \mathbb{R}^n,$$

which is a measurable set; hence $F_1(x, y)$ is measurable. Define $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $T(x, y) = (x - y, y)$, then T is a linear isomorphism. Then $F = F_1 \circ T$ is measurable, in fact, for any U open in $\mathbb{R}$, then $F_1^{-1}(U)$ is a measurable set, since T is a linear isomorphism, then $T^{-1}(F^{-1}(U))$ is also measurable, and thus $F^{-1}(U) = T^{-1}(F_1^{-1}(U))$ is measurable.

(b) If $F : \mathbb{R}^m \to \mathbb{R}$ and $G : \mathbb{R}^m \to \mathbb{R}$ are measurable, then $FG : \mathbb{R}^m \to \mathbb{R}$ is measurable.
The map $H : \mathbb{R}^2 \to \mathbb{R}$ defined by $H(x, y) = xy$ is a continuous map and $K(x) = (F(x), G(x))$ is measurable; hence the composition $H \circ K$ is measurable.

3. For $f \in L^p(\mathbb{T})$ and $h \in \mathbb{R}$, let $f_h(x) = f(x+h)$ denote the translation of f by h. If $1 \le p < \infty$, show that $f_h \to f$ in $L^p(\mathbb{T})$ as $h \to 0$. Give an example to show that this result is not true when $p = \infty$.

Proof. I will prove the statement of this problem in a more general setting i.e., on $L^p(\mathbb{R}^n)$. Obviously the result still holds $L^p(\mathbb{T})$.
If $f \in C_c(\mathbb{R}^n)$, then we can find a ball $B_R(0)$ such that f has support K in $B_R(0)$; hence f is uniformly continuous on $\mathbb{R}^n$.

Given $\epsilon > 0$, we can find $1 \gg \delta > 0$ such that whenever $\|x-y\| < \delta$, we have $|f(x)-f(y)| \le \epsilon$. Choose $|h| < \delta/2$, then for each $x \in \mathbb{R}^n$, $|f(x+h)-f(x)| < \epsilon$.

Hence whenever $|h| < \delta/2$,

$$\|\tau_h f - f\|_p^p = \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx \le \epsilon^p \mu(B_{R+1}(0)).$$

This implies f is L^p-continuous when $f \in C_c(\mathbb{R}^n)$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, then given $f \in L^p(\mathbb{R}^n)$, we can find $g \in C_c(\mathbb{R}^n)$ so that $\|f-g\|_p < \epsilon/3$.

Thus $\|\tau_h f - \tau_h g\|_p < \epsilon/3$ for each $h \in \mathbb{R}^n$. Since $g \in C_c(\mathbb{R})$, then g is L^p-continuous; hence we can find $\delta > 0$ so that $\|\tau_h g - g\|_p < \epsilon/3$ whenever $|h| < \delta$.

Therefore whenever $|h| < \delta$,

$$\|\tau_h f - f\|_p \le \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p < \epsilon,$$

which shows that f is L^p-continuous.

Example from Amanda

The contraexample that our problem is asking is the following:
Assume $f(x) \in L^\infty(\mathbb{T})$ is given by

$$f(x) := \chi_{[-\pi,0)}(x) = \begin{cases} 1, & -\pi \le x < 0 \\ 0, & 0 \le x \le \pi \end{cases}$$

Then for all $h \ne 0$ we have that $|f(x+h) - f(x)| = 1$ on $[-h, 0]$. By definition the norm on $L^\infty(\mathbb{T})$ is

$$\|f\|_\infty = \inf\{M |\, |f(x)| \le M \text{ a.e in } [a,b]\}$$

Since the measure of $[-h, 0]$ is h, this is a set of positive measure so we get that $\|f(x+h) - f(x)\|_\infty = 1$ for all $h \ne 0$. Therefore, $f(x+h) \not\to f(x)$.

$\square$

4. **a)** Compute the Fourier series expansion of

$$f(x) = |x| \qquad \text{for} |x| < \pi.$$

Proof. **a)** Let

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}.$$

Since $|x|$ is an even function (that means that $f(-x) = f(x)$), we get that $|x|$ has the following Fourier coefficients:

$$|x| = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_0 = \frac{1}{\pi}\int_{\mathbb{T}} |x|\, dx$$

$$a_n = \frac{1}{\pi}\int_{\mathbb{T}} |x| \cos(nx)\, dx.$$

Integrating, we obtain:

$$a_0 = \pi \qquad a_n = 0, \;\; \text{for}\, n \text{ even },$$

and also we get that

$$a_n = -\frac{4}{\pi n^2}, \;\; \text{for}\, n \text{ odd }.$$

$\square$

Proof. **b)**

Normalizing the Fourier coefficients we have computed above, we see that by **Parseval's theorem**:

$$\begin{aligned}
\int_{\mathbb{T}} |x|^2 &= \sum_{n\in\mathbb{Z}} |a_n|^2 \\
&= \frac{\pi^4}{2\pi} + \frac{16}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n=1}^{\infty}\frac{1}{16n^4}\right) \\
&= \frac{\pi^4}{2\pi} + \frac{15}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^4}.
\end{aligned}$$

Using that

$$\int_{\mathbb{T}} |x|^2 = \frac{2\pi^3}{3},$$

we get:

$$\frac{2\pi^3}{3} = \frac{\pi^4}{2\pi} + \frac{15}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^4},$$

which is equivalent to

$$\frac{2\pi^3}{3} - \frac{\pi^4}{2\pi} = \frac{15}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Symplifying, we obtain:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$\square$