1 Linear functionals on Hilbert spaces

- **Linear functionals.** A bounded linear function on a complex Hilbert space $\mathcal{H}$ is a bounded scalar-valued linear map $\phi : \mathcal{H} \to \mathbb{C}$. (We replace $\mathbb{C}$ by $\mathbb{R}$ for real spaces.)

- **Dual space.** The space of bounded linear functionals on $\mathcal{H}$ is the topological dual space of $\mathcal{H}$, denoted $\mathcal{H}^*$. The norm of $\phi : \mathcal{H} \to \mathbb{C}$ is
  \[ \|\phi\|_{\mathcal{H}^*} = \sup_{x \neq 0} \left( \frac{|\phi(x)|}{\|x\|} \right) = \sup_{\|x\|=1} |\phi(x)| \]

- **Reisz representation theorem.** If $\phi \in \mathcal{H}^*$ then there is a unique $x \in \mathcal{H}$ such that
  \[ \phi(y) = \langle x, y \rangle \quad \text{for every } y \in \mathcal{H}. \]

  The mapping $J : \mathcal{H}^* \to \mathcal{H}$ defined by $J : \phi \mapsto x$ is a conjugate-linear (i.e. $J(\lambda \phi) = \overline{\lambda} J \phi$) isometric isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$. Thus, using $J$, we may identify the dual space of a Hilbert space with the Hilbert space itself.

- **Weak convergence.** A sequence $\{x_n\}$ in $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$, written $x_n \rightharpoonup x$, if
  \[ \langle x_n, y \rangle \to \langle x, y \rangle \quad \text{for every } y \in \mathcal{H}. \]

- **Norm properties of weak convergence.** If $x_n \to x$ as $n \to \infty$, then $\{\|x_n\| : n \in \mathbb{N}\}$ is bounded and
  \[ \|x\| \leq \liminf_{n \to \infty} \|x_n\| \]
i.e. the norm is weakly lower semi-continuous. If
\[ x_n \to x \text{ and } \|x_n\| \to \|x\| \]
then \( x_n \to x \) strongly (in norm).

- **Necessary and sufficient condition for weak convergence.** Let \( D \) be a dense subset of a Hilbert space \( \mathcal{H} \). Then \( x_n \to x \) in \( \mathcal{H} \) if and only if \( \{\|x_n\|\} \) is bounded and
\[ \langle x_n, y \rangle \to \langle x, y \rangle \quad \text{for every } y \in D. \]

- **Banach-Alaoglu theorem.** The closed unit ball of a Hilbert space is weakly compact.

- **Minimization problems.** Let \( D \) be a weakly closed subset of a Hilbert space \( \mathcal{H} \). A real-valued function \( F : D \subset \mathcal{H} \to \mathbb{R} \) is weakly lower semi-continuous (wlsc) on \( D \) if
\[ F(x) \leq \liminf_{n \to \infty} F(x_n) \]
for all weakly convergent sequences \( \{x_n\} \) in \( D \), where \( x_n \to x \) as \( n \to \infty \). If \( D \) is weakly closed and bounded and \( F \) is wlsc on \( D \), then \( F \) is bounded from below and attains its infimum on \( D \).

2 Bounded linear operators on a Hilbert space

- **Bounded operators.** A linear operator \( A : \mathcal{H} \to \mathcal{K} \) between Hilbert spaces \( \mathcal{H}, \mathcal{K} \) is bounded if its operator norm
\[ \|A\| = \sup_{x \neq 0} \left( \frac{\|Ax\|_{\mathcal{K}}}{\|x\|_{\mathcal{H}}} \right) = \sup_{\|x\|_{\mathcal{H}} = 1} \|Ax\|_{\mathcal{K}} = \sup_{\|x\|_{\mathcal{H}} = 1, \|y\|_{\mathcal{K}} = 1} |\langle Ax, y \rangle_{\mathcal{K}}|. \]
is finite. The space of bounded linear maps from \( \mathcal{H}, \mathcal{K} \) is denoted \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \). It is a Banach space when equipped with the operator norm. If \( \mathcal{H} = \mathcal{K} \), we write \( \mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H}) \).

- **Adjoint.** The (Hilbert-space) adjoint of an operator \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is the bounded operator \( A^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \) such that
\[ \langle Ax, y \rangle_{\mathcal{K}} = \langle x, A^*y \rangle_{\mathcal{H}} \quad \text{for all } x \in \mathcal{H} \text{ and all } y \in \mathcal{K}. \]
• The algebra $B(H)$. The Banach space $B(H)$ is a $C^*$-algebra with respect to the composition product and the adjoint operation:

$$
\|AB\| \leq \|A\|\|B\|, \quad A^{**} = A, \quad (AB)^* = B^*A^*.
$$

The commutator of $A, B \in B(H)$ is the operator $[A, B] \in B(H)$ defined by $[A, B] = AB - BA$.

• Kernel-range theorem. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, then the kernel of $A$

$$
\ker A = \{x \in \mathcal{H} : Ax = 0\}
$$

is a closed linear subspace of $\mathcal{H}$, and the range of $A$

$$
\text{ran } A = \{y \in \mathcal{H} : y = Ax \text{ for some } x \in \mathcal{H}\}
$$

is a linear subspace of $\mathcal{H}$, which may or may not be closed. We always have

$$
\mathcal{H} = \text{ran } A \oplus \ker A^*.
$$

• Self-adjoint operators. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is self-adjoint if $A^* = A$, meaning that

$$
\langle Ax, y \rangle = \langle x, Ay \rangle \quad \text{for all } x, y \in \mathcal{H}.
$$

• Sesquilinear forms. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ defines a sesquilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ (meaning that $a$ is conjugate-linear in the first argument and linear in the second argument) by

$$
a(x, y) = \langle x, Ay \rangle.
$$

If $A$ is self-adjoint, then $a(x, y) = \overline{a(y, x)}$, $a(x, x) \in \mathbb{R}$, and

$$
\|A\| = \sup_{x \neq 0} \frac{\|\langle x, Ax \rangle\|}{\|x\|^2}.
$$

• Normal operators. A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is normal if $A^*, A$ commute, meaning that

$$
A^* A = AA^*.
$$

Self-adjoint and unitary operators on $\mathcal{H}$ are normal.
• **Unitary operators.** An operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is unitary if
\[
U^*U = I_{\mathcal{H}}, \quad UU^* = I_{\mathcal{K}}.
\]
In that case, $U$ maps any orthonormal basis of $\mathcal{H}$ to an orthonormal basis of $\mathcal{K}$, and preserves inner-products,
\[
\langle Ux, Uy \rangle_{\mathcal{K}} = \langle x, y \rangle_{\mathcal{H}} \quad \text{for all } x, y \in \mathcal{H},
\]
so $U$ defines an isometric isomorphism of $\mathcal{H}$ onto $\mathcal{K}$.

• **Orthogonal projections.** An orthogonal projection on a Hilbert space $\mathcal{H}$ is a bounded linear operator $P \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P$ (projection) and $P^* = P$ (self-adjoint or orthogonal).

• **Projection theorem.** Every orthogonal projection $P$ on $\mathcal{H}$ gives a direct sum decomposition
\[
\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{M} = \text{ran } P, \quad \mathcal{M}^\perp = \text{ker } P
\]
where $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$. Conversely, every closed subspace $\mathcal{M} \subset \mathcal{H}$ is associated with an orthogonal projection in this way.