1. If $1 \leq p < \infty$, show that the trigonometric polynomials are dense in $L^p(\mathbb{T})$.

2. For fixed $z \in \mathbb{C}$, let $J_n(z)$ denote the $n$th Fourier coefficient of the function $e^{iz\sin x}$, meaning that

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin x} e^{-inx} \, dx \quad \text{for } n \in \mathbb{Z}.$$ 

(a) What is $J_n(0)$? Show that $J_{-n}(z) = (-1)^n J_n(z)$.

(b) Derive the recurrence relations

$$\frac{2n}{z} J_n(z) = J_{n-1}(z) + J_{n+1}(z), \quad 2J'_n(z) = J_{n-1}(z) - J_{n+1}(z)$$

where the prime denotes a derivative with respect to $z$.

(c) Deduce from (b) that $J_n(z)$ is a solution of Bessel’s equation

$$z^2 J''_n + z J'_n + (z^2 - n^2) J_n = 0.$$ 

3. A family of (not necessarily positive) functions $\{\phi_n \in L^1(\mathbb{T}) : n \in \mathbb{N}\}$ is an approximate identity if:

- $\int \phi_n \, dx = 1$ for every $n \in \mathbb{N}$;
- $\int |\phi_n| \, dx \leq M$ for some constant $M$ and all $n \in \mathbb{N}$;
- $\lim_{n \to \infty} \int_{\delta < |x| < \pi} |\phi_n| \, dx = 0$ for every $\delta > 0$.

If $f \in L^1(\mathbb{T})$, show that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$.

4. (a) Let $\{a_n : n \geq 0\}$ be a sequence of non-negative real numbers such that $a_n \to 0$ as $n \to \infty$ and

$$a_{n+1} - 2a_n + a_{n-1} \geq 0.$$

Show that the series

$$\sum_{n=1}^{\infty} n (a_{n+1} - 2a_n + a_{n-1})$$
converges to $a_0$. HINT. $\sum (a_{n+1} - a_n)$ is a convergent, decreasing telescoping series.

(b) For $N \geq 0$, let $K_N \geq 0$ denote the Fejér kernel

$$K_N(x) = \sum_{n=-N}^{N} \left( 1 - \frac{|n|}{N+1} \right) e^{inx}.$$

Show that the series

$$f(x) = \sum_{n=1}^{\infty} n (a_{n+1} - 2a_n + a_{n-1}) K_{n-1}(x)$$

converges in $L^1(T)$ to a non-negative function $f \in L^1(T)$ whose Fourier coefficients are $a_{|n|}$ i.e.

$$f(x) \sim \sum_{n \in \mathbb{Z}} a_{|n|} e^{inx}.$$

(c) Show that there is a function $f \in L^1(T)$ such that

$$f(x) \sim \sum_{|n| \geq 2} \frac{1}{\log |n|} e^{inx}.$$

(d) Suppose that $f \in L^1(T)$ has imaginary Fourier coefficients $\{ib_n : n \in \mathbb{Z}\}$ such that $b_n \geq 0$ for $n \geq 0$ and $b_{-n} = -b_n$. Show that

$$\sum_{n=1}^{\infty} \frac{b_n}{n}$$

converges. HINT. The integral

$$F(x) = \int_{0}^{x} f(t) \, dt$$

is a continuous function (in fact, absolutely continuous) with Fourier coefficients

$$\frac{1}{2\pi} \int F(x) e^{-inx} \, dx = \frac{b_n}{in} \quad \text{for } n \neq 0.$$

Use the fact that $K_N \ast F(0)$ converges to $F(0)$ since $\{K_N\}$ is an approximate identity.

(e) Show that there is no function $f \in L^1(T)$ such that

$$f(x) \sim \sum_{|n| \geq 2} \frac{i \text{sgn } n}{\log |n|} e^{inx}.$$

(Here, sgn $n$ is equal to 1 if $n > 0$ and $-1$ if $n < 0$.)