Problem Set 3: Math 201B
Due: Friday, January 21

1. Suppose that $\sum_{n=0}^{\infty} c_n$ is a series of complex numbers with partial sums

$$s_n = \sum_{k=0}^{n} c_k.$$ 

The series is Borel summable with Borel sum $s$ if the following limit exists:

$$s = \lim_{x \to +\infty} e^{-x} \left( \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \right).$$

(a) If the series $\sum_{n=0}^{\infty} c_n = s$ is convergent, show that it is Borel summable with Borel sum equal to $s$ (meaning that Borel summation is regular).

(b) For what complex numbers $a \in \mathbb{C}$ is the geometric series

$$\sum_{n=0}^{\infty} a^n$$

Borel summable? What is its Borel sum? For what $a \in \mathbb{C}$ is this series Cesàro summable? Abel summable?

(c) Do you get anything useful from the Borel summation of a Fourier series?

2. Let $A(\mathbb{T})$ denote the space of integrable functions whose Fourier coefficients are absolutely convergent. That is, $f \in A(\mathbb{T})$ if

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty.$$ 

(a) If $f \in A(\mathbb{T})$, show that $f \in C(\mathbb{T})$. Also show that $f \in A(\mathbb{T})$ if and only if $f = g * h$ for some functions $g, h \in L^2(\mathbb{T})$.

(b) If $f, g \in A(\mathbb{T})$, show that $fg \in A(\mathbb{T})$ and express $\hat{fg}$ in terms of $\hat{f}, \hat{g}$.

Optional question!

(c) Give an example of a function $f \in C(\mathbb{T})$ such that $f \notin A(\mathbb{T})$. 

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3. Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ denote the unit disc in the complex plane. The Hardy space $H^2(D)$ is the space of functions with a power series expansion

$$ F(z) = \sum_{n=0}^{\infty} c_n z^n $$

such that

$$ \sum_{n=0}^{\infty} |c_n|^2 < \infty. $$

This is a Hilbert space with inner product

$$ \langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} \bar{a}_n b_n. $$

(a) If (2) holds, show that the power series (1) converges in $D$ to a holomorphic (analytic) function $F : D \to \mathbb{C}$. (You can use standard definitions and facts from complex analysis.)

(b) Is $1/(1 - z) \in H^2(D)$? If $\theta_0 \in \mathbb{T}$, give an example of a function $F \in H^2(D)$ which does not extend to a function that is analytic at $z = e^{i\theta_0}$.

(c) If $F \in H^2(D)$, show that

$$ \|F\|_{H^2}^2 = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| F(re^{i\theta}) \right|^2 d\theta < \infty. $$

Show conversely that if $F : D \to \mathbb{C}$ is a holomorphic function such that

$$ \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left| F(re^{i\theta}) \right|^2 d\theta < \infty $$

then $F \in H^2(D)$.

(d) Let

$$ \tilde{H}^2(\mathbb{T}) = \{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \}. $$

If $F \in H^2(D)$ is given by (1) and $0 < r < 1$, define $f_r \in L^2(\mathbb{T})$ by

$$ f_r(\theta) = F(re^{i\theta}). $$

Show that $f_r \to f$ as $r \to 1^-$ in $L^2(\mathbb{T})$ where

$$ f(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta} \in \tilde{H}^2(\mathbb{T}). $$

Conversely, if $f \in \tilde{H}^2(\mathbb{T})$, define $F : D \to \mathbb{C}$ by

$$ F(re^{i\theta}) = (P_r * f)(\theta) $$

where $P_r$ is the Poisson kernel. Show that $F \in H^2(\mathbb{T})$. 

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