1. Let $T^d = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}$ denote the $d$-dimensional Torus. For $s > 0$, let $H^s(T^d)$ denote the space of functions $f \in L^2(T^d)$ such that
\[
\sum_{n \in \mathbb{Z}^d} (1 + |n|^{2s}) \left| \hat{f}(n) \right|^2 < \infty.
\]
Prove that if $s > d/2$ and $f \in H^s(T^d)$, then $f \in C(T^d)$.

Remarks.

- Roughly speaking, this result means that the existence of more than one-half a weak $L^2$-derivative per space dimension implies that the function is continuous.

- Analogous Sobolev embedding theorems hold for $L^p$-derivatives, in which case at least $k > d/p$ weak $L^p$ derivatives (more than $1/p$ derivatives per space dimension) are required to imply continuity. When $p \neq 2$, however, a proof using Fourier analysis is not straightforward.

2. Suppose that $f \in L^1(T)$ is weakly differentiable and its weak derivative $f' = 0$ is zero. Prove that $f = \text{constant}$ (up to pointwise a.e. equivalence).

Remarks.

- The same result, with essentially the same proof, holds for distributions: If $T \in \mathcal{D}'(T)$ has zero distributional derivative, then $T$ is constant.
3. Define the principal-value functional $T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ by

\[
\langle T, \phi \rangle = \text{p.v.} \int_{\mathbb{T}} \cot \left( \frac{x}{2} \right) \phi(x) \, dx
\]

\[
= \lim_{\epsilon \to 0^+} \left( \int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) \cot \left( \frac{x}{2} \right) \phi(x) \, dx.
\]

(a) Show that $T \in \mathcal{D}'(\mathbb{T})$ is a well-defined periodic distribution.

(b) Compute the Fourier coefficients $\hat{T}(n)$ of $T$.

Remarks.

- The principal value distribution

\[
p.v. \frac{1}{\pi} \cot \left( \frac{x}{2} \right) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (-i \text{sgn } n) e^{inx}
\]

is the kernel of the periodic Hilbert transform

\[H : L^2(\mathbb{T}) \to L^2(\mathbb{T})\]

defined by the convolution

\[Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \cot \left( \frac{x - y}{2} \right) f(y) \, dy.
\]

The Fourier expression is

\[\hat{(Hf)}(n) = -i \text{sgn } n \hat{f}(n)\]

i.e. multiplication by $-i \text{sgn } n$. The Hilbert transform is a unitary map on the space of periodic $L^2$-functions with zero mean, and is a basic example of a singular integral operator.
4. (a) If $T \in \mathcal{D}'(\mathbb{T})$ is a periodic distribution, show that there exists an integer $k \geq 0$ and a constant $C$ such that

$$|\langle T, \phi \rangle| \leq C \|\phi\|_{C^k} \quad \text{for all } \phi \in \mathcal{D}(\mathbb{T}) \quad (1)$$

where

$$\|\phi\|_{C^k} = \sum_{j=0}^{k} \sup_{x \in \mathbb{T}} \left| \phi^{(j)}(x) \right|$$

denotes the $C^k$-norm of $\phi$.

Remarks.

- This result depends on the compactness of $\mathbb{T}$. An example of a distribution $T \in \mathcal{D}'(\mathbb{R})$ that does not have finite order is given by

$$T = \sum_{n=1}^{\infty} \delta^{(n)}(x - n),$$

meaning that $T$ is a sum of derivatives of $\delta$-functions whose order increases as their support moves further away from the origin.