FINAL: MATH 203B
Winter, 2007
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Instructions: There are two parts: attempt every question in Part I, and choose one question from Part II. Closed book. Give complete proofs. You may use any standard theorem provided you state it carefully. Good Luck!

PART I

Problem 1. For what values of $s \geq 0$ do the following functions

$$f(x) = \sum_{n \in \mathbb{Z}} \frac{e^{inx}}{\sqrt{1+n^2}}, \qquad g(x) = \sum_{n \in \mathbb{Z}} e^{inx - |n|},$$

belong to $H^s(\mathbb{T})$? What does the Sobolev imbedding theorem imply about the continuity and order of continuous differentiability of these functions?

Solution.

- Since
$$\sum_{n \in \mathbb{Z}} \frac{|n|^{2s}}{1+n^2} < \infty$$
for $s < 1/2$, we have $f \in H^s(\mathbb{T})$ for $s < 1/2$. This condition is not sufficient for the Sobolev imbedding theorem to imply anything about the continuity of f.

- Since
$$\sum_{n \in \mathbb{Z}} |n|^{2s} e^{-2|n|} < \infty$$
for every $s \geq 0$, we have $g \in H^s(\mathbb{T})$ for every $s \geq 0$. The Sobolev imbedding theorem implies that $g \in C^\infty(\mathbb{T})$.

Problem 2. Suppose that $f \in L^1(\mathbb{R})$, and for $n \in \mathbb{N}$ let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| > 1/n, \\ 0 & \text{if } |x| \leq 1/n. \end{cases}$$

Prove that $f_n \to f$ as $n \to \infty$ with respect to the L^1-norm.

Solution.

- For every $n \in \mathbb{N}$, we have

$$|f - f_n| \leq |f| \in L^1(\mathbb{R}).$$

  Since $|f - f_n| \to 0$ pointwise almost everywhere on $\mathbb{R}$ as $n \to \infty$, the Lebesgue dominated convergence theorem, implies that

$$\lim_{n\to\infty} \int |f - f_n|\, dx = \int \lim_{n\to\infty} |f - f_n|\, dx = 0,$$

  meaning that $f_n \to f$ in $L^1(\mathbb{R})$.

Problem 3. Suppose that $P : \mathcal{H} \to \mathcal{H}$ is a bounded, not necessarily self-adjoint, projection (meaning that $P^2 = P$) and $P \neq 0, I$. For $\lambda \in \mathbb{C}$, compute $(\lambda I - P)^{-1}$ explicitly when it exists, and show that the spectrum of P is the set $\{0, 1\}$. HINT. Consider the series expansion of $(\lambda I - P)^{-1}$.

Solution.

- For $0 < |\lambda| < 1$, we have

$$\begin{aligned}
(\lambda I - P)^{-1} &= \frac{1}{\lambda}\left(I - \frac{1}{\lambda}P\right)^{-1} \\
&= \frac{1}{\lambda}\left(I + \frac{1}{\lambda}P + \frac{1}{\lambda^2}P^2 + \frac{1}{\lambda^3}P^3 + \dots\right) \\
&= \frac{1}{\lambda}\left[I + \left(\frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} + \dots\right)P\right] \\
&= \frac{1}{\lambda}I + \frac{1}{\lambda(\lambda - 1)}P.
\end{aligned}$$

- By direct calculation, using $P^2 = P$, we have for every $\lambda \neq 0, 1$ that

$$(\lambda I - P)\left(\frac{1}{\lambda}I + \frac{1}{\lambda(\lambda-1)}P\right) = \left(\frac{1}{\lambda}I + \frac{1}{\lambda(\lambda-1)}P\right)(\lambda I - P) = I,$$

  which proves that $(\lambda I - P)$ is invertible and λ belongs to the resolvent set of P.

- Since $P \neq 0$, there exists $x \in \mathcal{H}$ such that $Px \neq 0$. Then $P(Px) = Px$, so $\lambda = 1$ is in the point spectrum of P.

- Since $P \neq I$, there exists $x \in \mathcal{H}$ such that $Px - x \neq 0$. Then

$$P(Px - x) = 0,$$

  so $\lambda = 0$ is in the point spectrum of P.

- It follows that $\sigma(P) = \{0, 1\}$.

Problem 4. For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ define

$$f_n(x) = \begin{cases} n^{-\alpha} e^{inx} & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n, \end{cases}$$

For what values of α does the sequence (f_n) converge in $L^2(\mathbb{R})$: (a) strongly; (b) weakly?

Solution.

- (a) We compute that

$$\|f_n\|^2 = 2n^{1-2\alpha}.$$

Hence, $f_n \to 0$ if $\alpha > 1/2$, while (f_n) is unbounded and therefore cannot converge strongly if $\alpha < 1/2$.

- If $\alpha = 1/2$, then for $m > n$

$$\|f_m - f_n\|^2 \geq 2(m-n)m^{-1} \to 2 \qquad \text{as } m \to \infty,$$

so the sequence is not Cauchy and does not converge strongly.

- The sequence converges strongly if and only $\alpha > 1/2$.

- (b) The sequence converges strongly and therefore weakly to 0 if $\alpha > 1/2$, and is unbounded and therefore does not converge weakly if $\alpha < 1/2$.

- If $\alpha = 1/2$, then an integration by parts shows that if $\phi \in C_c^\infty(\mathbb{R})$, then

$$\int_{\mathbb{R}} \phi(x) f_n(x)\, dx \to 0 \qquad \text{as } n \to \infty.$$

Since (f_n) is bounded and $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ it follows that $f_n \rightharpoonup 0$ as $n \to \infty$.

- The sequence converges weakly if and only if $\alpha \geq 1/2$.

Part II

Problem 5. Let $\{e_n \mid n \in \mathbb{N}\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$, and define $C \subset \mathcal{H}$ by

$$C = \left\{ x \in \mathcal{H} \mid \sum_{n\in\mathbb{N}} \left(1 + \frac{1}{n}\right)^2 |\langle e_n, x\rangle|^2 \le 1 \right\}.$$

(a) Prove that C is a closed, bounded, convex subset of $\mathcal{H}$. (Recall that C is convex if $tx + (1-t)y \in C$ whenever $x, y \in C$ and $t \in [0,1]$.)
(b) Prove that C has no element with greatest norm.

Solution.

- (a) If $x_k \in C$ and $x_k \to x$, then

$$\sum_{n\in\mathbb{N}} \left(1 + \frac{1}{n}\right)^2 |\langle e_n, x\rangle|^2 = \lim_{k\to\infty} \sum_{n\in\mathbb{N}} \left(1 + \frac{1}{n}\right)^2 |\langle e_n, x_k\rangle|^2 \le 1,$$

  so $x \in C$, and C is closed.

- If $x \in C$, then

$$\|x\|^2 = \sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 \le \sum_{n\in\mathbb{N}} \left(1 + \frac{1}{n}\right)^2 |\langle e_n, x\rangle|^2 \le 1,$$

  so C is bounded (and contained in the closed unit ball of $\mathcal{H}$).

- The function $s : \mathbb{R} \to \mathbb{R}$ defined by $s(x) = x^2$ is convex, meaning that if $x, y \in \mathbb{R}$ and $t \in [0,1]$,

$$s\left(tx + (1-t)y\right) \le ts(x) + (1-t)s(y).$$

  It follows that if $z = tx + (1-t)y$, with $x, y \in \mathcal{H}$ and $t \in [0,1]$, then

$$\begin{aligned} |\langle e_n, z\rangle|^2 &= |t\langle e_n, x\rangle + (1-t)\langle e_n, y\rangle|^2 \\ &\le \left(t\,|\langle e_n, x\rangle| + (1-t)\,|\langle e_n, y\rangle|\right)^2 \\ &\le t\,|\langle e_n, x\rangle|^2 + (1-t)\,|\langle e_n, y\rangle|^2 . \end{aligned}$$

Hence, if $x, y \in C$, then

$$\begin{aligned}\sum_{n\in\mathbb{N}}\left(1+\frac{1}{n}\right)^2|\langle e_n, z\rangle|^2 &\le t\sum_{n\in\mathbb{N}}\left(1+\frac{1}{n}\right)^2|\langle e_n, x\rangle|^2 \\ &\quad +(1-t)\sum_{n\in\mathbb{N}}\left(1+\frac{1}{n}\right)^2|\langle e_n, y\rangle|^2, \\ &\le 1,\end{aligned}$$

so $z \in C$, and C is convex.

- (b) We have seen that $\|x\| \le 1$ if $x \in C$. On the other hand,

$$x_n = \frac{1}{1+1/n}\, e_n \in C.$$

Since $\|x_n\| \to 1$ as $n \to \infty$, it follows that

$$\sup_{x\in C}\|x\| = 1.$$

- If $x \in C$ and $\|x\| = 1$, then

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\left\{\left(1+\frac{1}{n}\right)^2-1\right\}|\langle e_n, x\rangle|^2 \\ &\qquad = \sum_{n\in\mathbb{N}}\left(1+\frac{1}{n}\right)^2|\langle e_n, x\rangle|^2 - \|x\|^2 \\ &\qquad \le 0,\end{aligned}$$

which implies that $\langle e_n, x\rangle = 0$ for every $n \in \mathbb{N}$, or $x = 0$. This contradiction proves that C contains no element with norm equal to 1.

Problem 6. Let $U : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the unitary map from a complex sequence $c = (c_n)_{n\in\mathbb{Z}}$ to the function $f : \mathbb{T} \to \mathbb{C}$ whose sequence of Fourier coefficients is c:

$$(Uc)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Define the right-shift map $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by

$$Sc = b, \qquad \text{where } c = (c_n),\ b = (b_n), \text{ and } b_n = c_{n-1}.$$

Define the multiplication operator $M : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$(Mf)(x) = e^{ix} f(x).$$

(a) Show that $S = U^{-1}MU$.
(b) Determine the spectrum of M, and deduce the spectrum of S. Classify the spectrum.

Solution.

- (a) We have

  $$(US)c = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_{n-1}e^{inx} = e^{ix}\frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx} = (MU)c,$$

  which implies that $S = U^{-1}MU$.

- (b) If $|\lambda| \neq 1$, then $(M-\lambda I)$ is an invertible map on $L^2(\mathbb{T})$, with inverse

  $$(M - \lambda I)^{-1}f(x) = \frac{1}{e^{ix} - \lambda} f(x),$$

  $(e^{ix} - \lambda)^{-1}$ is a bounded function on $\mathbb{T}$.

- If $|\lambda| = 1$ then the range of the multiplication operator $e^{ix} - \lambda$ is a proper dense set of $L^2(\mathbb{T})$. Thus, the spectrum of M consists of the unit circle $\{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, and is entirely continuous.

- Since S is unitarily equivalent to M, it has the same spectrum.