# The Heat Equation

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## The heat equation on a circle

We consider the diffusion of heat in an insulated circular ring. We let $t \in [0, \infty)$ denote time and $x \in \mathbb{T}$ a spatial coordinate along the ring. After a suitable non-dimensionalization, the temperature $u(x,t)$ of the ring satisfies the following initial value problem:

$$\begin{aligned} u_t &= u_{xx} & x \in \mathbb{T}, \quad t > 0, \\ u(x,0) &= f(x) & x \in \mathbb{T}. \end{aligned}$$

Here, $f : \mathbb{T} \to \mathbb{R}$ is a given initial temperature distribution. The PDE $u_t = u_{xx}$ is called the heat, or diffusion equations; in several space-dimensions it is $u_t = \Delta u$.

Our aim is to solve for $u(x,t)$ when $t > 0$. In order to state a rigorous result, we first formulate the problem more precisely as an evolution equation in a Hilbert space.

Let $U(t) = u(\cdot, t)$ denote the temperature distribution at time $t \geq 0$. We assume this to be at least square-integrable, so that $U : [0, \infty) \to L^2(\mathbb{T})$. Thus, instead of regarding the temperature $u(x,t)$ as a scalar-valued function $u$ of two independent variables $(x,t)$, we regard it as a vector-valued function $U$ of a single variable $t$ whose value $U(t)$ is a function of $x$.

We then write the PDE $u_t = u_{xx}$ as an evolution equation for $U$ of the form

$$\frac{dU}{dt} = AU.$$

Here, as we will explain, the time-derivative $dU/dt$ is defined in an $L^2$-sense, and $A = \partial_x^2$.

First, consider the case of a function $U : [0, \infty) \to \mathcal{H}$ that takes values in a Hilbert space $\mathcal{H}$. We denote the norm on $\mathcal{H}$ by $\| \cdot \|$. We say that $U$ is continuous on $[0, \infty)$ if

$$\lim_{h \to 0} \|U(t+h) - U(t)\| = 0$$

for all $t \in [0, \infty)$, where the right-hand limit $h \to 0^+$ is understood if $t = 0$. We denote the space of such continuous functions by $C\left([0, \infty); \mathcal{H}\right)$.

We say that $U$ is continuously differentiable on $[0,\infty)$, with derivative

$$\frac{dU}{dt} = V \in C\left([0,\infty);\mathcal{H}\right),$$

if

$$\lim_{h\to 0}\left\|\frac{U(t+h)-U(t)}{h} - V(t)\right\| = 0$$

for all $t \in [0,\infty)$, where the right-hand limit $h \to 0^+$ is understood if $t = 0$. We denote the space of such continuously differentiable functions by $C^1\left([0,\infty);\mathcal{H}\right)$.

The PDE $u_t = u_{xx}$ implies that $U$ and its time-derivative $dU/dt$ 'live' in different spaces. In order to ensure that $dU/dt \in L^2(\mathbb{T})$, we will require that $U \in H^2(\mathbb{T})$. We define an unbounded linear operator $A$ on $L^2(\mathbb{T})$ by

$$A : H^2(\mathbb{T}) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T}), \qquad Af = \partial_x^2 f,$$

where the spatial derivative $\partial_x$ is understood in a weak sense. That is, if $f \in H^2(\mathbb{T})$ and $n \in \mathbb{Z}$, then

$$\widehat{(Af)}(n) = -n^2 \hat{f}(n),$$

where $\hat{f}(n)$ denotes the $n$th Fourier coefficient of $f$.

A precise formulation of the initial value problem (it is far from the only one) is then the following: Given $f \in H^2(\mathbb{T})$, solve

$$\frac{dU}{dt} = AU, \qquad U(0) = f, \tag{1}$$

$$U \in C\left([0,\infty); H^2(\mathbb{T})\right) \cap C^1\left([0,\infty); L^2(\mathbb{T})\right). \tag{2}$$

**Theorem.** There exists a unique solution of (1)–(2). This solution has the property that $U(t) \in C^\infty(\mathbb{T})$ for all $t > 0$. Moreover $U(t) \to \overline{f}$ as $t \to \infty$, where

$$\overline{f} = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)\,dx$$

is the mean of the the initial data over the circle $\mathbb{T}$; specifically,

$$\left\|U(t) - \overline{f}\right\| \leq e^{-t}\left\|f - \overline{f}\right\|.$$

**Proof.** First, we prove that a solution is unique. Suppose that $U_1$, $U_2$ are two solutions of (1)–(2). Then $U = U_1 - U_2$ satisfies (1)–(2) with $f = 0$. The continuity of the $L^2$-inner product $\langle \cdot, \cdot \rangle$ implies that $\|U\|^2 : [0, \infty) \to [0, \infty)$ is a differentiable function, and

$$\begin{aligned} \frac{d}{dt}\|U\|^2 &= \frac{d}{dt}\langle U, U \rangle \\ &= \left\langle \frac{dU}{dt}, U \right\rangle + \left\langle U, \frac{dU}{dt} \right\rangle \\ &= \langle AU, U \rangle + \langle U, AU \rangle . \end{aligned}$$

By Parseval's theorem,

$$\langle A, AU \rangle = -\sum_{n \in \mathbb{R}} n^2 \left| \hat{U}(n) \right|^2 \leq 0.$$

It follows that $\|U\|^2(t)$ satisfies, for $t \geq 0$,

$$\frac{d}{dt}\|U\|^2 \leq 0, \qquad \|U\|^2(0) = 0.$$

Gronwall's inequality then implies that $\|U\|^2(t) = 0$ for $t \geq 0$, so $U_1 = U_2$, and a solution is unique.

To prove the existence of a solution, we solve the equation by use of Fourier series to get

$$U(t)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{-n^2 t} e^{inx}.$$

One can verify directly that this defines a solution of (1)–(2).

If $t > 0$, then

$$\sum_{n \in \mathbb{Z}} n^{2k} e^{-2n^2 t} < \infty$$

for every $k \in \mathbb{N}$, so $U(t) \in H^k(\mathbb{T})$ for any $f \in L^2(\mathbb{T})$. The Sobolev imbedding theorem then implies that $U(t) \in C^\infty(\mathbb{T})$ whenever $t > 0$.

Finally, by Parseval's theorem,

$$\left\| U(t) - \overline{f} \right\|^2 = \left\| \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} \hat{f}(n) e^{-n^2 t} e^{inx} \right\|^2$$

$$
\begin{aligned}
&= \sum_{n\neq 0} \left|\hat{f}(n)\right|^2 e^{-2n^2 t} \\
&\leq e^{-2t} \sum_{n\neq 0} \left|\hat{f}(n)\right|^2 \\
&\leq e^{-2t} \left\|f - \overline{f}\right\|^2,
\end{aligned}
$$

which proves the result. □

## A diffusion semigroup

We may write the solution we have just obtained as

$$U(t) = T(t)f$$

where the bounded linear operator $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined, for $t \geq 0$, by

$$\widehat{T(t)f} = e^{-n^2 t}\hat{f}(n).$$

Using the convolution theorem, for $t > 0$ we may write

$$T(t)f = g^t * f$$

where the function $g^t \in C^\infty(\mathbb{T})$ is defined by

$$g^t(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx}.$$

The operator $T(t)$ maps the solution of (1)–(2) at time 0 to the solution at time $t$. As can be verified directly — because $e^{-n^2 s}e^{-n^2 t} = e^{-n^2(s+t)}$ — and as follows from the fact that the evolution of $U$ does not depend explicitly on $t$, the function

$$T : [0, \infty) \to \mathcal{B}\left(L^2(\mathbb{T})\right)$$

satisfies

$$
\begin{aligned}
&T(s)T(t) = T(s+t) \qquad \text{for } s, t \geq 0, \\
&T(0) = I.
\end{aligned}
$$

A family of operators $\{T(t) \mid t \geq 0\}$ with these properties is called a one-parameter semi-group.

For every $f \in L^2(\mathbb{T})$, the function $U(t) = T(t)f$ is a continuous function $U : [0, \infty) \to L^2(\mathbb{R})$, so $\{T(t) \mid t \geq 0\}$ is called a strongly continuous semi-group. Note, however, that $T(t)f$ is not differentiable with respect to $t$ at $t = 0^+$ unless $f \in H^2(\mathbb{T})$, which explains the restriction on $f$ in the previous section.

The solution of an $n \times n$ system of ODEs for $U : \mathbb{R} \to \mathbb{C}^n$,

$$\frac{dU}{dt} = AU, \qquad U(0) = f,$$

where $A : \mathbb{C}^n \to \mathbb{C}^n$, may be written as

$$U(t) = e^{tA} f.$$

Thus, we may regard $T(t)$ as providing a definition of the operator-exponential

$$T(t) = e^{t\partial_x^2}.$$

From this perspective, we have an example of an operator-valued function which satisfies the functional equation $T(s)T(t) = T(s + t)$. In the scalar-valued case, the solutions are the usual exponential functions $T(t) = e^{at}$.

One fundamental difference between the finite-dimensional case and the infinite-dimensional case considered here for the heat equation — where $T(t)$ is generated by an unbounded operator $A$ which has arbitrarily large negative eigenvalues, $\lambda_n = -n^2$ — is that $T(t)$ is defined only for $t \geq 0$. This fact is closely tied to the smoothing and irreversibility of the heat equation, and explains why the operators $T(t)$ form a semi-group, rather than a group. The inverse of $T(t)$ would be $T(-t)$ if the equation could be solved backward in time.