Midterm: Solutions
Math 201B, Winter 2007

Problem 1. Suppose that $\lambda \in \mathbb{C}$ and $\lambda \notin \mathbb{Z}$. Prove that for every $f \in L^2(\mathbb{T})$ there is a unique solution $u \in H^1(\mathbb{T})$ of the differential equation

$$iu' + \lambda u = f.$$

Solution.

- Computing Fourier coefficients, we see that $u \in L^2(\mathbb{T})$ is a solution if and only if
$$i(in)\hat{u}_n + \lambda \hat{u}_n = \hat{f}_n.$$
This equation has a unique solution
$$\hat{u}_n = \frac{\hat{f}_n}{\lambda - n},$$
which is well-defined since $\lambda \notin \mathbb{Z}$.

- For $\lambda \notin \mathbb{Z}$ there exists a constant C such that
$$\left| \frac{n}{\lambda - n} \right| \leq C \qquad \text{for all } n \in \mathbb{Z}.$$
It follows that if $f \in L^2(\mathbb{T})$, then
$$\sum_{n\in\mathbb{Z}} n^2 \left|\hat{u}_n\right|^2 = \sum_{n\in\mathbb{Z}} \frac{n^2 \left|\hat{f}_n\right|^2}{|\lambda - n|^2} \leq C^2 \sum_{n\in\mathbb{Z}} \left|\hat{f}_n\right|^2 = C^2 \|f\|_2^2 < \infty.$$
Hence, $u \in H^1(\mathbb{T})$.

Problem 2. If $f \in L^1(\mathbb{R})$, define a function $\hat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

Prove that $\hat{f} \in C_b(\mathbb{R})$, meaning that $\hat{f}$ is bounded and continuous.

Solution.

- We have

  $$\left|\hat{f}(\xi)\right| = \left|\int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx\right| \leq \int_{\mathbb{R}} \left|f(x) e^{-ix\xi}\right|\, dx = \int_{\mathbb{R}} |f(x)|\, dx = \|f\|_1,$$

  so $\hat{f}$ is bounded, with

  $$\|f\|_\infty \leq \|f\|_1.$$

- If $\xi \to \xi_0$, then, by the continuity of $e^{i\theta}$,

  $$f(x)e^{-ix\xi} \to f(x)e^{-ix\xi_0} \quad \text{pointwise for every } x \in \mathbb{R}.$$

  Moreover,

  $$\left|f(x)e^{-ix\xi}\right| \leq |f(x)| \in L^1(\mathbb{R}).$$

  The Lebesgue dominated convergence theorem therefore implies that

  $$\begin{aligned}
  \lim_{\xi \to \xi_0} \hat{f}(\xi) &= \lim_{\xi \to \xi_0} \int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx \\
  &= \int_{\mathbb{R}} \lim_{\xi \to \xi_0} \left[f(x)e^{-ix\xi}\right]\, dx \\
  &= \int_{\mathbb{R}} f(x)e^{-ix\xi_0}\, dx \\
  &= \hat{f}(\xi_0),
  \end{aligned}$$

  so $\hat{f}$ is continuous.

Remark. The function $\hat{f}$ is the Fourier transform of f. In fact, $\hat{f} \in C_0(\mathbb{R})$ if $f \in L^1(\mathbb{R})$, meaning that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. This result, the Riemann-Lebesgue lemma, follows from the density of $C_c^\infty(\mathbb{R})$ in $L^1(\mathbb{R})$.

Problem 3. Suppose that $\{\phi_n \in L^2(\mathbb{R}) \mid n \in \mathbb{N}\}$ is an orthonormal set of functions in $L^2(\mathbb{R})$. For $m, n \in \mathbb{N}$, define $\phi_{m,n} : \mathbb{R}^2 \to \mathbb{C}$ by

$$\phi_{m,n}(x, y) = \phi_m(x)\phi_n(y).$$

Prove that $\{\phi_{m,n} \in L^2(\mathbb{R}^2) \mid m, n \in \mathbb{N}\}$ is an orthonormal set in $L^2(\mathbb{R}^2)$.

Solution.

- We have

$$\begin{aligned} \langle \phi_{j,k}, \phi_{m,n} \rangle &= \int_{\mathbb{R}^2} \overline{\phi}_{j,k}(x, y)\phi_{m,n}(x, y)\, dxdy \\ &= \int_{\mathbb{R}^2} \overline{\phi}_j(x)\phi_m(x)\overline{\phi}_k(y)\phi_n(y)\, dxdy. \end{aligned}$$

- Using Fubini's theorem for non-negative functions, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} \left|\overline{\phi}_j(x)\phi_m(x)\overline{\phi}_k(y)\phi_n(y)\right|\, dxdy \\ &\qquad = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|\overline{\phi}_j(x)\phi_m(x)\overline{\phi}_k(y)\phi_n(y)\right|\, dx\right) dy \\ &\qquad = \left(\int_{\mathbb{R}} \left|\overline{\phi}_j(x)\phi_m(x)\right|\, dx\right)\left(\int_{\mathbb{R}} \left|\overline{\phi}_k(y)\phi_n(y)\right|\, dy\right) \end{aligned}$$

  By the Cauchy-Schwartz inequality, and the normalization of the ϕ_n,

$$\int_{\mathbb{R}} \left|\overline{\phi}_j(x)\phi_m(x)\right|\, dx \le \left(\int_{\mathbb{R}} |\phi_j(x)|^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} |\phi_m(x)|^2\, dx\right)^{1/2} = 1.$$

  Hence, $\phi_{j,k}\phi_{m,n} \in L^1(\mathbb{R}^2)$, and we can apply Fubini's theorem.

- Using Fubini's theorem and the orthonormality of the ϕ_n in $L^2(\mathbb{R})$, we compute that

$$\begin{aligned} \langle \phi_{j,k}, \phi_{m,n} \rangle &= \int_{\mathbb{R}^2} \overline{\phi}_j(x)\phi_m(x)\overline{\phi}_k(y)\phi_n(y)\, dxdy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{\phi}_j(x)\phi_m(x)\overline{\phi}_k(y)\phi_n(y)\, dx\right) dy \\ &= \delta_{j,m} \int_{\mathbb{R}} \overline{\phi}_k(y)\phi_n(y)\, dy \\ &= \delta_{j,m}\delta_{k,m}, \end{aligned}$$

  which shows that the $\phi_{m,n}$ are orthonormal in $L^2(\mathbb{R}^2)$.

Problem 4. Suppose that $f, g \in L^2(\mathbb{T})$. Prove that

$$\|f * g\|_\infty \leq \|f\|_2 \, \|g\|_2,$$

where $f * g$ denotes the convolution of f, g and

$$\|f\|_\infty = \sup_{x \in \mathbb{T}} |f(x)|, \qquad \|f\|_2 = \left(\int_\mathbb{T} |f(x)|^2 \, dx \right)^{1/2}.$$

Prove that $f * g \in C(\mathbb{T})$ is continuous.

Solution.

- By the Cauchy-Schwartz inequality,

$$\begin{aligned} |(f * g)(x)| &= \left| \int_\mathbb{T} f(x-y) g(y) \, dy \right| \\ &\leq \left(\int_\mathbb{T} |f(x-y)|^2 \, dy \right)^{1/2} \left(\int_\mathbb{T} |g(y)|^2 \, dy \right)^{1/2} \\ &= \|f\|_2 \|g\|_2. \end{aligned}$$

  Taking the supremum of this inequality over $x \in \mathbb{T}$, we get the result.

- Since the trigonometric polynomials are dense in $L^2(\mathbb{T})$, there are sequences (p_n), (q_n) of trigonometric polynomials such that

$$\|p_n - f\|_2 \to 0, \quad \|q_n - g\|_2 \to 0 \qquad \text{as } n \to \infty.$$

  It follows that

$$\begin{aligned} \|(f * g) - (p_n * q_n)\|_\infty &\leq \|(f - p_n) * g\|_\infty + \|p_n * (g - q_n)\|_\infty \\ &\leq \|f - p_n\|_2 \, \|g\|_2 + \|p_n\|_2 \, \|g - q_n\|_2 \\ &\to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

  Hence, $f * g$ is the uniform limit of the continuous functions $p_n * q_n$, so $f * g$ is continuous.

Problem 5. Let M be the linear space of complex sequences (x_n) of the form $(x_1, x_2, x_3, \ldots, x_N, 0, 0, \ldots)$ where $x_n \in \mathbb{C}$, $x_n = 0$ for $n > N$, for some $N \in \mathbb{N}$, and

$$\sum_{n=1}^{N} x_n = 0.$$

What is the closure $\overline{M}$ of M in $\ell^2(\mathbb{N})$? What is the orthogonal complement $M^\perp$ of M in $\ell^2(\mathbb{N})$?

Solution.

- The closure of M is the whole space $\ell^2(\mathbb{N})$, so $M^\perp = \{0\}$.
- To show that M is dense in $\ell^2(\mathbb{N})$, suppose that

  $$y = (y_1, y_2, y_3, \ldots, y_N, 0, 0, \ldots)$$

  is any terminating complex sequence, with

  $$\sum_{n=1}^{N} y_n = c.$$

  For any $K \in \mathbb{N}$, define $x = (x_n)$ by

  $$x_n = \begin{cases} y_n & \text{if } 1 \le n \le N, \\ -c/K & \text{if } N+1 \le n \le N+K, \\ 0 & \text{if } n > N+K. \end{cases}$$

  Then

  $$\sum_{n=1}^{N+K} x_n = 0,$$

  so $x \in M$.
- We compute that

  $$\|x - y\|^2 = K\frac{|c|^2}{K^2} = \frac{|c|^2}{K} \to 0 \quad \text{as } K \to \infty.$$

  Therefore, the closure of M contains all terminating sequences. Since the terminating sequences are dense in $\ell^2(\mathbb{N})$, we have $\overline{M} = \ell^2(\mathbb{N})$.