FINAL: MATH 203B

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Instructions: Closed book. You may use any standard theorem, provided you state it carefully. Good Luck!

Problem 1. Which of the following Fourier series define a square-integrable function on $\mathbb{T}$? Say as much as you can about the continuity and differentiability of the sums.

$$f_1(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^4} e^{inx},$$

$$f_2(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^4} e^{i(|n|!)x},$$

$$f_3(x) = \sum_{n=-\infty}^{\infty} \frac{1}{\log(1+n^4)} e^{inx},$$

$$f_4(x) = \sum_{n=-\infty}^{\infty} e^{-(1+n^4)} e^{inx}.$$

Problem 2. Let $\mathcal{H}$ be the Hilbert space of functions $f : [-1, 1] \to \mathbb{C}$ such that

$$\int_{-1}^{1} \frac{|f(x)|^2}{\sqrt{1-x^2}}\, dx < \infty,$$

with the inner-product

$$\langle f, g\rangle = \int_{-1}^{1} \frac{\overline{f(x)} g(x)}{\sqrt{1-x^2}}\, dx.$$

Show that the Chebyshev polynomials,

$$T_n(x) = \cos(n\theta), \qquad \text{where } \cos\theta = x \text{ and } 0 \le \theta \le \pi,$$

$n = 0, 1, 2, \ldots$, form an orthogonal set in $\mathcal{H}$, and

$$\|T_0\| = \sqrt{\pi}, \qquad \|T_n\| = \sqrt{\frac{\pi}{2}} \quad n \geq 1.$$

(Hint: $2 \cos m\theta \cos n\theta = \cos(m + n)\theta + \cos(m - n)\theta$.) Given that $\{T_n \mid n = 0, 1, \ldots\}$ is complete, write out the expansion of $f \in \mathcal{H}$ with respect to the Chebyshev polynomials, and say explicitly in what sense the series converges.

Problem 3. The Dirichlet problem for Laplace's equation in the unit disc for $u(r, \theta)$ is

$$\begin{aligned} &\frac{1}{r}\left(r u_r\right)_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad \text{in } r < 1 \\ &u(1, \theta) = f(\theta), \end{aligned}$$

where $f : \mathbb{T} \to \mathbb{R}$ is a given function, and (r, θ) are polar coordinates. Solve this problem by expansion of u in Fourier series,

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta}.$$

(Hint: try $u_n(r) = r^k$ for suitable k.) Show that your solution can be written in $r < 1$ as

$$u(r, \theta) = (g_r * f)(\theta),$$

where $*$ denotes the convolution on $\mathbb{T}$ and $g_r : \mathbb{T} \to \mathbb{R}$ is given by

$$g_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

Problem 4. Let $\Delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the discrete Laplacian operator defined by $\Delta(x_n) = (y_n)$ where

$$y_n = x_{n+1} - 2x_n + x_{n-1}.$$

Determine the spectrum of Δ, and classify its spectrum into its discrete, continuous, and residual parts.

Problem 5. A linear operator $K : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is compact if it maps bounded sets into precompact sets. Show that K is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.