FINAL: MATH 203B
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Instructions: Closed book. Give complete proofs. You may use any standard theorem provided you state it carefully. Good Luck!

Problem 1. (a) Define the Fourier coefficients and Fourier series of a 2π-periodic function $f \in L^2(\mathbb{T})$.
(b) Compute the Fourier coefficients of the 2π-periodic function such that

$$f(x) = x, \qquad |x| < \pi.$$

For what real values of $s \geq 0$ is it true that $f \in H^s(\mathbb{T})$?
(c) State Parseval's theorem, and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 2. Use Fourier series to solve the linearized KdV equation,

$$\begin{aligned} u_t &= u_{xxx}, \\ u(x,0) &= f(x), \end{aligned}$$

where $u(\cdot, t) \in L^2(\mathbb{T})$. Show that the solution operators form a strongly-continuous one-parameter unitary group in $L^2(\mathbb{T})$.

Problem 3. Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be a *reducing subspace* for A if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant subspaces of A, meaning that $A(\mathcal{M}) \subset \mathcal{M}$ and $A(\mathcal{M}^\perp) \subset \mathcal{M}^\perp$. Show that $\mathcal{M}$ is a reducing subspace for A if and only if $PA = AP$ where P is the orthogonal projection onto $\mathcal{M}$.

Problem 4. (a) Define weak convergence in a Hilbert space.

(b) Let $\{\mathbf{e}_n \mid n \in \mathbb{N}\}$ be an orthonormal set in a Hilbert space, and define

$$\mathbf{x}_n = [1 + (-1)^n]\,\mathbf{e}_n, \quad \mathbf{y}_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}\mathbf{e}_k, \quad \mathbf{z}_n = (-1)^n \sum_{k=1}^{n} \frac{1}{k}\mathbf{e}_k.$$

For each of the sequences $(\mathbf{x}_n)$, $(\mathbf{y}_n)$, $(\mathbf{z}_n)$ say if it converges weakly and find the weak limit if it exists.

Problem 5. Let $K : L^2(0,1) \to L^2(0,1)$ be the integral operator defined by

$$Ku(x) = \int_0^1 e^{x-y} u(y)\, dy.$$

(a) Find the range of K. Is the range of K closed? Is K compact?

(b) Compute the adjoint operator K^*, and find its kernel.

(c) Verify explicitly that $Ku = f$ is solvable if and only if $f \perp \ker K^*$.

Problem 6. State the spectral theorem for compact self-adjoint operators. If A is a compact self-adjoint operator with nonnegative eigenvalues, prove that there is an operator B such that $B^2 = A$.