Sample Midterm Problems

Math 201B, Winter 2007

Problem 1. Let $E \subset \mathbb{R}$ be a measurable subset of $\mathbb{R}$. Define a linear subspace M of $L^2(\mathbb{R})$ by

$$M = \left\{ f \in L^2(\mathbb{R}) \mid f(x) = 0 \text{ a.e. in } E \right\}.$$

Find $M^\perp$. What is the orthogonal projection of $f \in L^2(\mathbb{R})$ onto M?

Problem 2. Let

$$E = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < \infty, \quad 0 < y < 1 \right\}.$$

Prove that

$$\int_E y \sin x e^{-xy} \, dx dy = \frac{1}{2} \log 2.$$

Problem 3. Let $T, S \in L^2(\mathbb{T})$ be the triangular and square waves, defined respectively by

$$T(x) = |x| \quad \text{if } |x| < \pi, \qquad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x \leq 0, \end{cases}$$

Compute the Fourier coefficients of T, S. Show that $T \in H^1(\mathbb{T})$ and $T' = S$. Show that $S \notin H^1(\mathbb{T})$.

Problem 4. Suppose that the Fourier coefficients $\hat{f}_n$ of a function $f \in L^1(\mathbb{T})$ satisfy $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, meaning that

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}_n \right| < \infty.$$

Prove that f is a continuous function.

Problem 5. An indexed set of vectors $\{u_\alpha \mid \alpha \in A\}$ in a Hilbert space $\mathcal{H}$ is said to be *stable* if there exist constants $m, M > 0$ such that for all

$$\{c_\alpha \mid c_\alpha \in \mathbb{C}, \alpha \in A\} \in \ell^2(A)$$

we have

$$m \sum_{\alpha \in A} |c_\alpha|^2 \leq \left\| \sum_{\alpha \in A} c_\alpha u_\alpha \right\|^2 \leq M \sum_{\alpha \in A} |c_\alpha|^2 .$$

(a) Show that a stable set is linearly independent, and an orthonormal set is stable.

(b) Suppose that $\{u_\alpha \mid \alpha \in A\}$ is a set of normalized vectors ($\|u_\alpha\| = 1$) such that

$$\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 < 1.$$

Show that $\{u_\alpha \mid \alpha \in A\}$ is stable.

(c) Let $\{e_n \mid n \in \mathbb{Z}\}$ be an orthonormal set, and define

$$u_n = \frac{e_n + e_{n+1}}{\sqrt{2}}.$$

Show that $\{u_n \mid n \in \mathbb{Z}\}$ is *not* stable.