Sample Midterm Problems
Brief Solutions
Math 201B, Winter 2007

Problem 1. Let $E \subset \mathbb{R}$ be a measurable subset of $\mathbb{R}$. Define a linear subspace M of $L^2(\mathbb{R})$ by

$$M = \left\{ f \in L^2(\mathbb{R}) \mid f(x) = 0 \text{ a.e. in } E \right\}.$$

Find $M^\perp$. What is the orthogonal projection of $f \in L^2(\mathbb{R})$ onto M?

Solution.

- Let $E^c = \mathbb{R} \setminus E$ be the complement of E. Then

$$M^\perp = \left\{ f \in L^2(\mathbb{R}) \mid f(x) = 0 \text{ a.e. in } E^c \right\}.$$

- The direct-sum decomposition of f is

$$f = \chi_{E^c} f + \chi_E f,$$

  where χ_E is the characteristic function of E, and $\chi_{E^c} f$ is the orthogonal projection of f onto M.

Problem 2. Let

$$E = \left\{ (x, y) \in \mathbb{R}^2 \mid 0 < x < \infty, \quad 0 < y < 1 \right\}.$$

Prove that

$$\int_E y \sin x e^{-xy}\, dxdy = \frac{1}{2} \log 2.$$

Solution.

- To apply Fubini's theorem, first check that

$$\begin{aligned} \int_E \left| y \sin x e^{-xy} \right| dxdy &\le \int_E y e^{-xy}\, dxdy \\ &= \int_0^1 \left(\int_0^\infty y e^{-xy}\, dx \right) dy \\ &< \infty. \end{aligned}$$

- Then compute the iterated integral

$$\int_0^1 \left(\int_0^\infty y \sin x e^{-xy} \, dx \right) dy.$$

Problem 3. Let $T, S \in L^2(\mathbb{T})$ be the triangular and square waves, defined respectively by

$$T(x) = |x| \quad \text{if } |x| < \pi, \qquad S(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ -1 & \text{if } -\pi < x \leq 0, \end{cases}$$

Compute the Fourier coefficients of T, S. Show that $T \in H^1(\mathbb{T})$ and $T' = S$. Show that $S \notin H^1(\mathbb{T})$.

Solution.

- From the definition of the Fourier coefficient,

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx,$$

  we compute that

$$\begin{aligned} \hat{T}_n &= \sqrt{\frac{2}{\pi}} \left[\frac{(-1)^n - 1}{n^2} \right] \quad n \neq 0, \qquad \hat{T}_0 = \frac{\pi^2}{\sqrt{2\pi}}, \\ \hat{S}_n &= i\sqrt{\frac{2}{\pi}} \left[\frac{(-1)^n - 1}{n} \right] \quad n \neq 0, \qquad \hat{S}_0 = 0. \end{aligned}$$

- The sum

$$\sum_{n \in \mathbb{Z}} n^2 \left| \hat{T}_n \right|^2$$

  converges, since $\sum 1/n^2$ converges, so $T \in H^1(\mathbb{T})$. Since $\hat{S}_n = in\hat{T}_n$, we have $S = T'$.

- The series

$$\sum_{n \in \mathbb{Z}} n^2 \left| \hat{S}_n \right|^2$$

  does not converge, since the terms do not approach zero as $n \to \infty$, so $S \notin H^1(\mathbb{T})$.

Problem 4. Suppose that the Fourier coefficients $\hat{f}_n$ of a function $f \in L^1(\mathbb{T})$ satisfy $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, meaning that

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}_n \right| < \infty.$$

Prove that f is a continuous function.

Solution.

- Since $|\hat{f}_n e^{inx}| \leq |\hat{f}_n|$, the Fourier series

$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{f}_n e^{inx}$$

  converges uniformly by the Weierstrass M-test, so the sum g is continuous.

- The Fourier coefficients of g are equal to those of f, since in computing the Fourier coefficients of g we may exchange the order of integration and uniform limits.

- From Problem 2 in Problem Set 5, it follows that $f = g$, so f is continuous.

Problem 5. An indexed set of vectors $\{u_\alpha \mid \alpha \in A\}$ in a Hilbert space $\mathcal{H}$ is said to be *stable* if there exist constants $m, M > 0$ such that for all

$$\{c_\alpha \mid c_\alpha \in \mathbb{C}, \alpha \in A\} \in \ell^2(A)$$

we have

$$m \sum_{\alpha \in A} |c_\alpha|^2 \leq \left\| \sum_{\alpha \in A} c_\alpha u_\alpha \right\|^2 \leq M \sum_{\alpha \in A} |c_\alpha|^2 .$$

(a) Show that a stable set is linearly independent, and an orthonormal set is stable.

(b) Suppose that $\{u_\alpha \mid \alpha \in A\}$ is a set of normalized vectors ($\|u_\alpha\| = 1$) such that

$$\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 < 1.$$

Show that $\{u_\alpha \mid \alpha \in A\}$ is stable.

(c) Let $\{e_n \mid n \in \mathbb{Z}\}$ be an orthonormal set, and define

$$u_n = \frac{e_n + e_{n+1}}{\sqrt{2}}.$$

Show that $\{u_n \mid n \in \mathbb{Z}\}$ is *not* stable.

Solution.

- (a) If $\{u_\alpha \mid \alpha \in A\}$ is stable and

  $$\sum_{\alpha \in A} c_\alpha u_\alpha = 0,$$

  then

  $$\sum_{\alpha \in A} |c_\alpha|^2 = 0.$$

  Hence $c_\alpha = 0$ for all $\alpha \in A$, so a stable set is linearly independent. (In fact, what we have shown is stronger than linear independence, since we did not consider only finite linear combinations of the vectors.)

- For an orthonormal set

  $$\left\| \sum_{\alpha \in A} c_\alpha u_\alpha \right\|^2 = \sum_{\alpha \in A} |c_\alpha|^2$$

  so the set is stable, with $m = M = 1$.

- (b) We have

$$\left\| \sum_{\alpha \in A} c_\alpha u_\alpha \right\|^2 = \sum_{\alpha,\beta \in A} \overline{c_\alpha} c_\beta \left\langle u_\alpha, u_\beta \right\rangle .$$

  Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_{\alpha,\beta \in A} \overline{c_\alpha} c_\beta \left\langle u_\alpha, u_\beta \right\rangle &= \sum_{\alpha \in A} |c_\alpha|^2 + \sum_{\alpha \neq \beta} \overline{c_\alpha} c_\beta \left\langle u_\alpha, u_\beta \right\rangle \\
&\leq \sum_{\alpha \in A} |c_\alpha|^2 + \left(\sum_{\alpha \neq \beta} |\overline{c_\alpha} c_\beta|^2 \right)^{1/2} \left(\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 \right)^{1/2} \\
&\leq \sum_{\alpha \in A} |c_\alpha|^2 + \sum_{\alpha \in A} |c_\alpha|^2 \left(\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 \right)^{1/2} \\
&\leq 2 \sum_{\alpha \in A} |c_\alpha|^2,
\end{aligned}$$

  and

$$\begin{aligned}
\sum_{\alpha,\beta \in A} \overline{c_\alpha} c_\beta \left\langle u_\alpha, u_\beta \right\rangle &= \sum_{\alpha \in A} |c_\alpha|^2 + \sum_{\alpha \neq \beta} \overline{c_\alpha} c_\beta \left\langle u_\alpha, u_\beta \right\rangle \\
&\geq \sum_{\alpha \in A} |c_\alpha|^2 - \left(\sum_{\alpha \neq \beta} |\overline{c_\alpha} c_\beta|^2 \right)^{1/2} \left(\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 \right)^{1/2} \\
&\geq \left[1 - \left(\sum_{\alpha \neq \beta} |\langle u_\alpha, u_\beta \rangle|^2 \right)^{1/2} \right] \sum_{\alpha \in A} |c_\alpha|^2.
\end{aligned}$$

- (c) Consider, for example, $c_n = (-1)^n$ for $1 \leq n \leq N$ and $c_n = 0$ otherwise. Then

$$\sum_{n \in \mathbb{Z}} |c_n|^2 = N,$$

  and

$$\left\| \sum_{n \in \mathbb{Z}} c_n u_n \right\|^2 = \left\| \frac{(-1)^N e_{N+1} - e_1}{\sqrt{2}} \right\|^2 = 1.$$

  Since $N \in \mathbb{N}$ is arbitrary, there is no constant $m > 0$ with the required property for stability.