Problem Set 1
Math 201B, Winter 2007
Due: Friday, Jan 12

Problem 1. Suppose that X is a linear space with inner product $(\cdot, \cdot)$. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, prove that $(x_n, y_n) \to (x, y)$ as $n \to \infty$.

Problem 2. (a) Consider the linear space $C([0,1])$ equipped with the L^1-norm,

$$\|f\|_1 = \int_0^1 |f(x)| \, dx.$$

Prove that there is no inner product $(\cdot, \cdot)$ on $C([0,1])$ such that

$$\|f\|_1 = \sqrt{(f, f)}.$$

(b) Suppose that X is a normed linear space (over $\mathbb{C}$) whose norm $\|\cdot\|$ satisfies the parallelogram law. Define $(\cdot, \cdot) : X \times X \to \mathbb{C}$ by

$$(x, y) = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right\}.$$

Prove that $(\cdot, \cdot)$ is an inner product on X such that $\|x\| = \sqrt{(x, x)}$.

Problem 3. Let M be a linear subspace of a Hilbert space $\mathcal{H}$. Prove that $M^{\perp\perp} = \overline{M}$.

Problem 4. Consider $C([0,1])$ equipped with the sup-norm, and define the closed linear subspace

$$M = \left\{ g \in C([0,1]) \mid g(0) = 0, \int_0^1 g(x) \, dx = 0 \right\}.$$

Let $f \in C([0,1]) \setminus M$ be the function $f(x) = x$. Prove that

$$d(f, M) = \inf_{g \in M} \|f - g\|_\infty = \frac{1}{2},$$

but that the infimum is not attained for any $g \in M$. (Meaning that there is no “closest” element to f in M.)

Problem 5. We denote the Hölder semi-norm with exponent $1/2$ and the L^2-norm of a function $f : [0,1] \to \mathbb{R}$ by

$$[f] = \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x - y|^{1/2}}, \qquad \|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

We denote the sup-norm of f by $\|f\|_\infty$.

(a) If f is continuously differentiable on $[0,1]$, with derivative f', prove that

$$[f] \leq \|f'\|_2.$$

HINT. Cauchy-Schwarz.

(b) Given $R > 0$, let

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuously differentiable, } \|f\|_2 \leq R, \ \|f'\|_2 \leq R\}$$

Prove that $\mathcal{F}$ is a precompact subset of $C([0,1])$ equipped with the sup-norm.