Problem Set 2
Math 201B, Winter 2007
Due: Friday, Jan 19

Problem 1. Let $(x_n)_{n=1}^{\infty}$ be a seqence in a Banach space. Prove that the unordered sum

$$\sum_{n\in\mathbb{N}} x_n$$

converges if and only if the series

$$\sum_{n=1}^{\infty} x_n$$

converges unconditionally.

Problem 2. Suppose that the unordered sums

$$\sum_{\alpha\in A} x_\alpha, \qquad \sum_{\beta\in B} y_\beta$$

converge in a Hilbert space. Prove that

$$\left\langle \sum_{\alpha\in A} x_\alpha, \sum_{\beta\in B} y_\beta \right\rangle = \sum_{(\alpha,\beta)\in A\times B} \langle x_\alpha, y_\beta\rangle .$$

Problem 3. Let $\{e_\alpha \mid \alpha \in A\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Define

$$\mathcal{M} = \left\{ \sum_{\alpha\in A} c_\alpha e_\alpha \mid c_\alpha \in \mathbb{C},\ \sum_{\alpha\in A} |c_\alpha|^2 < \infty \right\}.$$

Prove that $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$.