Problem Set 3
Math 201B, Winter 2007
Due: Friday, Jan 26

Problem 1. Prove that an infinite-dimensional Hilbert space is a separable metric space if and only if it has a countable orthonormal basis.

Problem 2. Prove that if M is a dense linear subspace of a separable Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis consisting of elements in M.

Problem 3. Define the Legendre polynomials P_n by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

(a) Compute the first four Legendre polynomials, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$.
(b) Show that the Legendre polynomials are orthogonal in $L^2([-1, 1])$.
(c) Show that the Legendre polynomials are obtained by Gram-Schmidt orthogonalization of the monomials $\{1, x, x^2, \ldots\}$ in $L^2([-1, 1])$.
Hint. Show that $P_n(x)$ is orthogonal to x^m for $m < n$.
(d) Show that

$$\int_{-1}^{1} P_n(x)^2 \, dx = \frac{2}{2n+1}.$$

(e) Show that the Legendre polynomial P_n is an eigenfunction of the differential operator

$$L = -\frac{d}{dx} \left(1 - x^2\right) \frac{d}{dx}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n.$$

(f) Compute the polynomial $q(x)$ of degree 2 that is 'closest' to e^x on $[-1, 1]$, in the sense that

$$\int_{-1}^{1} \left|e^x - q(x)\right|^2 \, dx = \min \left\{ \int_{-1}^{1} \left|e^x - f(x)\right|^2 \, dx \mid f(x) = ax^2 + bx + c \right\}.$$

Problem 4. Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right).$$

(a) Define

$$\phi_n(x) = e^{-x^2/2} H_n(x)$$

Show that $\{\phi_n \mid n = 0, 1, 2, \ldots\}$ is an orthogonal set in $L^2(\mathbb{R})$.

(b) Show that the nth Hermite function ϕ_n is an eigenfunction of the linear operator

$$H = -\frac{d^2}{dx^2} + x^2$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

Hint: Let

$$A = \frac{d}{dx} + x, \qquad A^* = -\frac{d}{dx} + x.$$

Show that

$$A\phi_n = 2n\phi_{n-1}, \quad A^*\phi_n = \phi_{n+1}, \quad H = AA^* - 1.$$

Remark. In quantum mechanics, H arises as the Hamiltonian operator of a simple harmonic oscillator, and ϕ_n is an eigenstate of the oscillator with energy proportional to λ_n. The operators A^* and A are called creation and annhilation operators: A^* maps a state with n-quanta to one with $(n+1)$-quanta, while A maps a state with n-quanta to one with $(n-1)$-quanta.

Richard P. Feynman, Nobel Lecture, December 11, 1965:

> I remember that when someone had started to teach me about creation and annihilation operators, that this operator creates an electron, I said, "how do you create an electron? It disagrees with the conservation of charge", and in that way, I blocked my mind from learning a very practical scheme of calculation.