Problem Set 4
Math 201B, Winter 2007
Due: Friday, Feb 2

Problem 1. (a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Use the monotone convergence theorem to show that $f \in L^1(\mathbb{R})$.
(b) Suppose that $\{r_n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ is an enumeration of the rational numbers. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n),$$

where f is the function defined in (a). Show that $g \in L^1(\mathbb{R})$, even though it is unbounded on every interval.

Problem 2. If $f \in L^1(\mathbb{R})$, prove that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f \, dx = 0.$$

Give an example to show that this result need not be true if f is not integrable on $\mathbb{R}$.

Problem 3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1/x^2 & \text{if } 0 < y < x < 1, \\ -1/y^2 & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following integrals:

$$\int_{\mathbb{R}^2} |f(x, y)| \, dxdy; \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, dx \right) dy; \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, dy \right) dx.$$

Are your results consistent with Fubini's theorem?

Problem 4. Define $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$f(x, y) = xe^{-x^2\left(1+y^2\right)}.$$

Compute the iterated integrals with respect to x, y and y, x, and use Fubini's theorem to show that

$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

Problem 5. In a normed space X, let

$$B_r(a) = \{x \in X \mid \|x - a\| < r\}.$$

be the ball of radius $r > 0$ centered at $a \in X$. Lebesgue measure m on $\mathbb{R}^d$ (with the Euclidean norm, say) has the properties that every ball is measurable with finite, nonzero measure, and the measure of a ball is invariant under translations. That is, for every $0 < r < \infty$ and $a \in X$

$$0 < m\left(B_r(a)\right) < \infty, \qquad m\left(B_r(a)\right) = m\left(B_r(0)\right).$$

Prove that it is not possible to define a measure with these properties on an infinite-dimensional Hilbert space.

Hint. Consider a countably infinite collection of balls whose centers form an orthonormal set.

Remark. There is no analog of the translation invariant Lebesgue measure on infinite-dimensional linear spaces, which is a fundamental complication in the definition of the 'functional integrals' that arise in quantum field theory and the study of stochastic processes. As Yu. I. Manin comments in *Mathematics and Physics*, the nonexistence of such a measure in the limit of infinite-dimensions is a consequence of the concentration of the volume of a ball near its surface in high spatial dimensions:

> Almost two-thirds of a twenty-dimensional watermelon with radius 20 cm consists of rind if the rind is 1 cm thick.