Problem Set 5
Math 201B, Winter 2007
Due: Friday, Feb 9

Problem 1. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(x) = x^2 \qquad \text{for } -\pi \le x \le \pi.$$

(a) Compute the Fourier coefficients of f.
(b) Use Parseval's theorem to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Problem 2. Suppose that $(\phi_n)_{n=1}^{\infty}$ is an approximate identity on $\mathbb{T}$ and $f \in L^1(\mathbb{T})$.
(a) Prove that for every $n \in \mathbb{N}$

$$\|\phi_n * f\|_1 \le \|f\|_1.$$

(b) Prove that

$$\|\phi_n * f - f\|_1 \to 0 \qquad \text{as } n \to \infty.$$

Hint. $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$.
(c) If $f, g \in L^1(\mathbb{T})$ have the same Fourier coefficients, prove that $f = g$.

Problem 3. Consider the differential equation

$$-u'' + u = f.$$

(a) If $f \in L^2(\mathbb{T})$, use Fourier series to show that there is unique solution $u \in H^2(\mathbb{T})$.

(b) Show that $u = G * f$ for a suitable function G (called the Green's function).

(c) Show that $G \in H^s(\mathbb{T})$ for $s < 3/2$.

Problem 4. Suppose that $f : \mathbb{T} \to \mathbb{C}$ is continuous, with Fourier coefficients

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx} \, dx.$$

Let f_N denote the mean of the first $(N+1)$ partial sums of the Fourier series of f, meaning that

$$f_N(x) = \frac{1}{N+1} \sum_{M=0}^{N} \left(\frac{1}{\sqrt{2\pi}} \sum_{m=-M}^{M} \hat{f}_m e^{imx} \right).$$

(a) Show that $f_N = K_N * f$ where $K_N : \mathbb{T} \to \mathbb{R}$ is the Fejér kernel, given by

$$K_N(x) = \frac{1}{2\pi} \frac{1}{N+1} \sum_{n=-N}^{N} \left(N + 1 - |n|\right) e^{inx}.$$

(b) Show that $K_N(x)$ may also be written as

$$\begin{aligned} K_N(x) &= \frac{1}{2\pi} \frac{1}{N+1} \left[\frac{\sin\left((N+1)\, x/2\right)}{\sin\left(x/2\right)} \right]^2 \qquad x \neq 0, \\ K_N(0) &= \frac{1}{2\pi}(N+1). \end{aligned}$$

(c) Show that K_N is an approximate identity. What can you say about the convergence of f_N to f as $N \to \infty$?