Problem Set 6
Math 201B, Winter 2007
Due: Friday, Feb 23

Problem 1. Consider the Schrödinger equation on the circle,

$$\begin{aligned} iu_t &= u_{xx}, && x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) &= f(x), && x \in \mathbb{T}, \end{aligned}$$

where $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $f : \mathbb{T} \to \mathbb{C}$ and the derivatives are interpreted in an appropriate sense.

(a) Solve for $u(x,t)$ by the use of Fourier series. If $U(t) = u(\cdot, t) \in L^2(\mathbb{T})$, show that

$$U(t) = T(t)f$$

where $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear operator, defined for all $t \in \mathbb{R}$.

(b) Show that $T(t)$ is a unitary operator, meaning that for all $f, g \in L^2(\mathbb{T})$,

$$\langle T(t)f, T(t)g \rangle = \langle f, g \rangle.$$

(c) Briefly compare the qualitative properties (smoothing, reversiblility, long-time behavior) of the Schrödinger equation with those of the heat equation.

Problem 2. (a) Suppose that P, Q are orthogonal projections on a Hilbert space. Prove that $PQ = 0$ if and only if $\operatorname{ran} P \perp \operatorname{ran} Q$.

(b) Suppose that $\{P_1, P_2, \ldots, P_n\}$ is a family of orthogonal projections on a Hilbert space, and $P_j P_k = 0$ for $j \neq k$. Prove that $P_1 + P_2 + \ldots + P_n$ is an orthogonal projection.

(c) Suppose that $\{P_k \mid k \in \mathbb{N}\}$ is a countably-infinite family of orthogonal projections on a Hilbert space $\mathcal{H}$ such that

$$\bigoplus_{k \in \mathbb{N}} \operatorname{ran} P_k = \mathcal{H}, \qquad P_j P_k = 0 \quad \text{for } j \neq k.$$

Prove that for every $x \in \mathcal{H}$

$$\sum_{k=1}^{\infty} P_k x = x,$$

where the series converges strongly (i.e. with respect to the norm) in $\mathcal{H}$. Is it true or false that

$$\sum_{k=1}^{\infty} P_k = I,$$

where the series converges with respect to the operator norm on $\mathcal{B}(\mathcal{H})$?

Problem 3. (a) Suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces. Define $\mathcal{H}_1 \oplus \mathcal{H}_2$ as the linear space of ordered pairs

$$\mathcal{H}_1 \oplus \mathcal{H}_2 = \{(x_1, x_2) \mid x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2\},$$

with the inner product of $x, y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, with $x = (x_1, x_2)$, $y = (y_1, y_2)$, defined by

$$\langle x, y \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}.$$

Prove that $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a Hilbert space.

(b) Suppose that $\{\mathcal{H}_\alpha \mid \alpha \in A\}$ is an arbitrary indexed family of Hilbert spaces. Define

$$\bigoplus_{\alpha \in A} \mathcal{H}_\alpha = \left\{ (x_\alpha)_{\alpha \in A} \mid x_\alpha \in \mathcal{H}_\alpha, \quad \sum_{\alpha \in A} \|x_\alpha\|^2 < \infty \right\},$$

with the inner product of

$$x = (x_\alpha) \in \bigoplus_{\alpha \in A} \mathcal{H}_\alpha, \qquad y = (y_\alpha) \in \bigoplus_{\alpha \in A} \mathcal{H}_\alpha$$

defined by

$$\langle x, y \rangle = \sum_{\alpha \in A} \langle x_\alpha, y_\alpha \rangle.$$

Prove that $\bigoplus_{\alpha \in A} \mathcal{H}_\alpha$ is a Hilbert space.