Problem Set 7
Math 203B, Winter 2007

Problem 1. Suppose that $m : [0, 1] \to \mathbb{C}$ is a continuous complex-valued function on $[0, 1]$. Define a multiplication operator

$$M : L^2([0, 1]) \to L^2([0, 1])$$

by

$$(Mf)(x) = m(x)f(x).$$

(a) Prove that M is a bounded linear operator on $L^2([0, 1])$ and compute its adjoint M^*.

(b) For what functions m is M: (i) self-adjoint; (ii) skew-adjoint; (iii) unitary?

Problem 2. The Hilbert transform $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by

$$H\left(\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}\right) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} i \operatorname{sgn} n\, \hat{f}(n)e^{inx},$$

where

$$\operatorname{sgn} n = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n < 0. \end{cases}$$

That is, the Hilbert transform acts on a function by multiplying its nth Fourier coefficient by i if n is positive and $-i$ if n is negative.

(a) If $n \in \mathbb{N}$ is a positive integer, compute $H(\cos nx)$ and $H(\sin nx)$. Show that H is a bounded linear map on $L^2(\mathbb{T})$ and compute its norm.

(b) Show that H is skew-adjoint.

(c) Let $\mathcal{M}$ be the subspace of periodic functions with zero mean,

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} f\, dx = 0 \right\}.$$

Show that the range of H is equal to $\mathcal{M}$. What is the kernel of H?

(d) Show that $H^2 = -I$ on $\mathcal{M}$ and that H is a unitary transformation on $\mathcal{M}$.

Problem 3. Let $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ be the Hilbert spaces of periodic square-integrable functions and functions with square-integrable weak derivatives, respectively, with the inner products

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{T}} \overline{f} g \, dx, \qquad \langle f, g \rangle_{H^1} = \int_{\mathbb{T}} \left(\overline{f} g + \overline{f'} g' \right) \, dx.$$

Let $D : H^1(\mathbb{T}) \to L^2(\mathbb{T})$ be the derivative operator $D = d/dx$ defined by

$$\widehat{(Df)}(n) = in\widehat{f}(n).$$

Prove that $D^* : L^2(\mathbb{T}) \to H^1(\mathbb{T})$ is given by

$$D^* = D \left(D^2 - 1 \right)^{-1}.$$