Problem Set 8
Math 201B, Winter 2007
Due: Friday, March 9

Problem 1. Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map whose matrix with respect to the standard basis on $\mathbb{C}^2$ is

$$[H] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Why must e^{-itH} be unitary for any $t \in \mathbb{R}$? Compute the matrix of e^{-itH} and verify explicitly that it is unitary.

Problem 2. Define the right and left shift operators S and T on $\ell^2(\mathbb{N})$ by

$$\begin{aligned} S(x_1, x_2, x_3, x_4, \dots) &= (0, x_1, x_2, x_3, \dots), \\ T(x_1, x_2, x_3, x_4, \dots) &= (x_2, x_3, x_4, x_5, \dots). \end{aligned}$$

(a) Show that $\langle Sx, Sy \rangle = \langle x, y \rangle$ for all $x, y \in \ell^2(\mathbb{N})$. Is S a unitary map on $\ell^2(\mathbb{N})$?
(b) Show that $S^* = T$.
(c) Determine the range and kernel of S and T. Show that both operators have closed range and verify explicitly that

$$\ell^2(\mathbb{N}) = \operatorname{ran} S \oplus \ker S^* = \operatorname{ran} T \oplus \ker T^*.$$

(d) Given any $y \in \ell^2(\mathbb{N})$, find all solutions $x \in \ell^2(\mathbb{N})$, if any, of the equations: (i) $Sx = y$; (ii) $Tx = y$. Do S, T satisfy the Fredholm alternative?

Problem 3. For $n \in \mathbb{N}$, define the following functions $f_n, g_n, h_n : \mathbb{R} \to \mathbb{R}$:

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases}$$

$$h_n(x) = \begin{cases} 1 & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that none of the sequences (f_n), (g_n), (h_n) converge strongly in $L^2(\mathbb{R})$. Which sequences converge weakly? HINT. $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.