Solutions: Problem Set 1
Math 201B, Winter 2007

Problem 1. Suppose that X is a linear space with inner product $(\cdot, \cdot)$. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, prove that $(x_n, y_n) \to (x, y)$ as $n \to \infty$.

Solution.

- Using the Cauchy-Schwarz and triangle inequalities, we have

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n - x, y_n)| + |(x, y_n - y)| \\ &\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\| \\ &\leq \|x_n - x\| \, (\|y_n - y\| + \|y\|) + \|x\| \, \|y_n - y\| \\ &\leq \|x_n - x\| \, \|y_n - y\| + \|x_n - x\| \, \|y\| + \|x\| \, \|y_n - y\| \, . \end{aligned}$$

Since $\|x_n - x\| \to 0$, $\|y_n - y\| \to 0$ as $n \to \infty$, we see that

$$(x_n, y_n) \to (x, y) \qquad \text{as } n \to \infty.$$

Problem 2. (a) Consider the linear space $C([0,1])$ equipped with the L^1-norm,

$$\|f\|_1 = \int_0^1 |f(x)|\,dx.$$

Prove that there is no inner product $(\cdot,\cdot)$ on $C([0,1])$ such that

$$\|f\|_1 = \sqrt{(f,f)}.$$

(b) Suppose that X is a normed linear space (over $\mathbb{C}$) whose norm $\|\cdot\|$ satisfies the parallelogram law. Define $(\cdot,\cdot) : X \times X \to \mathbb{C}$ by

$$(x,y) = \frac{1}{4}\left\{\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right\}.$$

Prove that $(\cdot,\cdot)$ is an inner product on X such that $\|x\| = \sqrt{(x,x)}$.

Solution.

- (a) Consider, for example, $f(x) = 1$ and $g(x) = 2x$. Then

  $$\|f\|_1 = 1, \qquad \|g\|_1 = 1,$$

  while

  $$\|f-g\|_1 = \int_0^1 |1-2x|\,dx = \frac{1}{2},$$
  $$\|f+g\|_1 = \int_0^1 (1+2x)\,dx = 2.$$

  Thus

  $$\|f-g\|_1^2 + \|f+g\|_1^2 = \frac{17}{4}, \qquad 2\left(\|f\|_1^2 + \|g\|_1^2\right) = 4,$$

  so the norm does not satisfy the parallelogram law. Hence it is not obtained from an inner product.

- (b) Note that we may write the expression for $(\cdot,\cdot)$ as

  $$(x,y) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\left\|x + i^k y\right\|^2.$$

- It follows immediately from the definition that

$$(x,x) = \|x\|^2 \geq 0, \qquad (y,x) = \overline{(x,y)}.$$

  So the main thing we need to prove is that (x,y) is linear in y.

- Using the parallelogram law, we find for any $x, y, x \in X$ that

$$\begin{aligned}\left\|x+y+2i^k z\right\|^2 &+ \|x-y\|^2 \\ &= \left\|\left(x+i^k z\right)+\left(y+i^k z\right)\right\|^2 + \left\|\left(x+i^k z\right)-\left(y+i^k z\right)\right\|^2 \\ &= 2\left\|x+i^k z\right\|^2 + 2\left\|y+i^k z\right\|^2 .\end{aligned}$$

  Multiplying this equation by i^{-k}, summing the result over $0 \leq k \leq 3$, using the fact that $\sum_{k=0}^{3} i^{-k} = 0$, and using the definition of $(\cdot,\cdot)$, we get

$$(x+y, 2z) = 2(x,z) + 2(y,z).$$

- Setting $y = 0$ in this equation, and using the fact immediate from the definition that $(0,z) = 0$, we get $(x, 2z) = 2(x,z)$. It then follows that

$$(x+y, z) = (x,z) + (y,z) \,.$$

  Since $(y,x) = \overline{(x,y)}$, we also get

$$(x, y+z) = (x,y) + (x,z) \,.$$

- Repeated application of this result implies that for any $m \in \mathbb{N}$,

$$(x, my) = m\,(x,y) \,.$$

  It follows that for any $m, n \in \mathbb{N}$,

$$n\left(x, \frac{m}{n} y\right) = (x, my) = m\,(x,y) \,,$$

  so

$$\left(x, \frac{m}{n} y\right) = \frac{m}{n}\,(x,y) \,.$$

- Finally, using the density of the rationals in the reals, the continuity of the norm, and the immediate properties

  $$(x, -y) = -(x, y), \qquad (x, iy) = i(x, y),$$

  we conclude that

  $$(x, \lambda y) = \lambda\,(x, y)$$

  for all $\lambda \in \mathbb{C}$ This proves that $(\cdot, \cdot)$ defines an inner product.

Remark. According to P. Lax in *Functional Analysis*, this observation is due to von Neumann.

Problem 3. Let M be a linear subspace of a Hilbert space $\mathcal{H}$. Prove that $M^{\perp\perp} = \overline{M}$.

Solution.

- If $x \in M$ then $\langle x, y\rangle = 0$ for all $y \in M^{\perp}$, so $x \perp M^{\perp}$. It follows that $M \subset M^{\perp\perp}$.

- Since $M^{\perp\perp}$ is an orthogonal complement, it is a closed linear subspace, so
$$\overline{M} \subset M^{\perp\perp}.$$

- If $x \in \mathcal{H}$ then, by the projection theorem, we may write $x = y+z$ where $y \in \overline{M}$, $z \in M^{\perp}$. If $x \in M^{\perp\perp}$, then $\langle x, z\rangle = 0$, so $\langle y, z\rangle + \langle z, z\rangle = 0$. Since $M^{\perp} = \overline{M}^{\perp}$, we have $\langle y, z\rangle = 0$, so $\langle z, z\rangle = 0$. Hence $z = 0$ and $x = y \in \overline{M}$. It follows that
$$M^{\perp\perp} \subset \overline{M}.$$

- Combining these results, we get $M^{\perp\perp} = \overline{M}$.

Remark. The same argument shows that if $E \subset \mathcal{H}$ is any subset, then $E^{\perp\perp} = [E]$, where $[E]$ is the closed linear span of E.

Problem 4. Consider $C([0,1])$ equipped with the sup-norm, and define the closed linear subspace

$$M = \left\{ g \in C([0,1]) \mid g(0) = 0, \int_0^1 g(x)\, dx = 0 \right\}.$$

Let $f \in C([0,1]) \setminus M$ be the function $f(x) = x$. Prove that

$$d(f, M) = \inf_{g \in M} \|f - g\|_\infty = \frac{1}{2},$$

but that the infimum is not attained for any $g \in M$. (Meaning that there is no "closest" element to f in M.)

Solution.

- We consider real-valued functions for simplicity.
- We have for all $x \in [0,1]$ that

  $$f(x) - g(x) \leq \|f - g\|_\infty.$$

  Integrating this equation with respect to x, we get

  $$\int_0^1 f(x)\, dx - \int_0^1 g(x)\, dx \leq \|f - g\|_\infty$$

  Hence, if $f(x) = x$ and $g \in M$ so $\int_0^1 g(x)\, dx = 0$, then

  $$\frac{1}{2} \leq \|f - g\|_\infty.$$

  It follows that $d(f, M) \geq 1/2$.

- For sufficiently small $\epsilon > 0$, define the function

  $$g^\epsilon(x) = \begin{cases} -kx & \text{if } 0 \leq x \leq \delta, \\ x - 1/2 - \epsilon & \text{if } \delta < x \leq 1, \end{cases}$$

  where we choose $-k\delta = \delta - 1/2 - \epsilon$ to ensure the continuity of g^ϵ at $x = \delta > 0$ and

  $$\frac{1}{2} k\delta \left(\frac{1}{2} + \epsilon \right) = \frac{1}{2} \left(\frac{1}{2} - \epsilon \right)^2$$

to ensure that the integral of g^ϵ is zero. Explicitly,

$$\delta = \frac{2\epsilon}{1/2+\epsilon}, \qquad k = \frac{(1/2-\epsilon)^2}{2\epsilon}.$$

Then $g^\epsilon \in M$ and $\|f - g^\epsilon\|_\infty = 1/2 + \epsilon \to 1/2$ as $\epsilon \to 0^+$.

- It follows that $d(f, M) \leq 1/2$, so $d(f, M) = 1/2$.
- Suppose, for contradiction, that the infimum is attained, and $g \in M$ is such that $\|f - g\|_\infty = 1/2$. Then

$$g(x) - f(x) \geq -\frac{1}{2} \qquad \text{for all } x \in [0, 1].$$

Thus, writing

$$h(x) = g(x) - x + \frac{1}{2},$$

we see that $h \geq 0$.

- Since $\int_0^1 g(x)\,dx = 0$, we have

$$\int_0^1 h(x)\,dx = 0.$$

However, since $g(0) = 0$, we have $h(0) = 1/2$. Since h is continuous there is an interval $[0, \delta]$ of width $\delta > 0$ on which $h \geq 1/4$. Hence, since h is nonnegative,

$$\int_0^1 h(x)\,dx \geq \int_0^\delta h(x)\,dx \geq \frac{\delta}{4} > 0.$$

This contradiction proves that the infimum is not attained.

Problem 5. We denote the Hölder semi-norm with exponent $1/2$ and the L^2-norm of a function $f : [0,1] \to \mathbb{R}$ by

$$[f] = \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^{1/2}}, \qquad \|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx \right)^{1/2}.$$

We denote the sup-norm of f by $\|f\|_\infty$.

(a) If f is continuously differentiable on $[0,1]$, with derivative f', prove that

$$[f] \leq \|f'\|_2.$$

(b) Given $R > 0$, let

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuously differentiable, } \|f\|_2 \leq R,\ \|f'\|_2 \leq R\}$$

Prove that $\mathcal{F}$ is a precompact subset of $C([0,1])$ equipped with the sup-norm.

Solution.

- (a) By the fundamental theorem of calculus,

  $$f(x) - f(y) = \int_y^x f'(t) \, dt.$$

  The Cauchy-Schwarz inequality implies that

  $$\begin{aligned} \left| \int_y^x f'(t) \, dt \right| &= \left| \int_y^x 1 \cdot f'(t) \, dt \right| \\ &\leq \left(\int_y^x 1 \, dt \right)^{1/2} \left(\int_y^x [f'(t)]^2 \, dt \right)^{1/2} \\ &\leq |x-y|^{1/2} \left(\int_0^1 [f'(t)]^2 \, dt \right)^{1/2} \\ &\leq |x-y|^{1/2} \|f'\|_2. \end{aligned}$$

  Hence for all $0 \leq x \neq y \leq 1$, we have

  $$\frac{|f(x) - f(y)|}{|x-y|^{1/2}} \leq \|f'\|_2.$$

  Taking the supremum over $x \neq y$, we get $[f] \leq \|f'\|_2$.

- (b) Given $\epsilon > 0$, let $\delta = \epsilon^2/R^2$. If $f \in \mathcal{F}$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$, so the family $\mathcal{F}$ is equicontinuous.

- It follows from (a) that if $f \in \mathcal{F}$, then

$$|f(x) - f(y)| \leq R$$

  for all $x, y \in [0,1]$. Hence,

$$|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq R + |f(y)|.$$

  Integrating this equation over $[0,1]$ with respect to y, we get for every $x \in [0,1]$ that

$$|f(x)| \leq R + \int_0^1 |f(y)|\, dy.$$

  By the Cauchy-Schwarz inequality if $f \in \mathcal{F}$, then

$$\int_0^1 |f(y)|\, dy = \int_0^1 1 \cdot |f(y)|\, dy \leq \left(\int_0^1 1^2\, dy \right)^{1/2} \left(\int_0^1 |f(y)|^2\, dy \right)^{1/2} \leq R.$$

  Thus, $|f(x)| \leq 2R$, so $\|f\|_\infty \leq 2R$ and $\mathcal{F}$ is bounded.

- The Arzelà-Ascoli theorem implies that $\mathcal{F}$ is a precompact subset of $C([0,1])$.