**Solutions: Problem Set 2**
**Math 201B, Winter 2007**

**Problem 1.** Let $(x_n)_{n=1}^{\infty}$ be a sequence in a Banach space. Prove that the unordered sum

$$\sum_{n\in\mathbb{N}} x_n$$

converges if and only if the series

$$\sum_{n=1}^{\infty} x_n$$

converges unconditionally.

**Solution.**

- First, suppose that the unordered sum converges in the normed space $X$, to $x \in X$ say. Let $\sum_{m=1}^{\infty} y_m$ be a rearrangement of the series $\sum_{n=1}^{\infty} x_n$, so that $y_m = x_{\sigma(m)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is one-to-one and onto.
- Given $\epsilon > 0$, there is a finite subset $I \subset \mathbb{N}$ such that if $J \supset I$ is a finite subset of $\mathbb{N}$, then
$$\left\| \sum_{n\in J} x_n - x \right\| < \epsilon.$$
If
$$M \geq \max\left\{\sigma^{-1}(n) \mid n \in I\right\},$$
then $J = \{\sigma(1), \sigma(2), \ldots, \sigma(M)\} \supset I$. Hence,
$$\left\| \sum_{m=1}^{M} y_m - x \right\| = \left\| \sum_{n\in J} x_n - x \right\| < \epsilon,$$
so $\sum_{m=1}^{\infty} y_m = x$, and the series converges unconditionally.
- Conversely, suppose that the unordered sum does not converge. Then for each $x \in X$, there exists $\epsilon > 0$ such that for every finite subset $I \subset \mathbb{N}$ there exists a finite subset $J \supset I$ with
$$\left\| \sum_{n\in J} x_n - x \right\| \geq \epsilon.$$

Pick a finite set $J_1 \supset \{1\}$ with this property, and, given a finite set $J_N \subset \mathbb{N}$, pick a finite set $J_{N+1} \supset \{1, \ldots, N+1\} \cup J_N$ with this property. We then define a rearrangement of $\mathbb{N}$ by listing the elements of $J_1$ (in any order), followed by the elements of $J_2 \setminus J_1$ (if any), $J_3 \setminus J_2$, and so on. The resulting rearranged series has partial sums that differ infinitely often by at least $\epsilon$ from $x$. Hence, the series cannot converge unconditionally to any $x \in X$.

**Problem 2.** Suppose that the unordered sums

$$\sum_{\alpha \in A} x_\alpha, \qquad \sum_{\beta \in B} y_\beta$$

converge in a Hilbert space. Prove that

$$\left\langle \sum_{\alpha \in A} x_\alpha, \sum_{\beta \in B} y_\beta \right\rangle = \sum_{(\alpha,\beta) \in A \times B} \langle x_\alpha, y_\beta \rangle .$$

**Solution.**

- We write
$$x = \sum_{\alpha \in A} x_\alpha, \qquad y = \sum_{\beta \in B} y_\beta.$$
We want to show that the unordered sum of complex numbers
$$\sum_{(\alpha,\beta) \in A \times B} \langle x_\alpha, y_\beta \rangle$$
converges and is equal to $\langle x, y \rangle$.

- First we give a simple, but incomplete, argument. If $\epsilon > 0$, then there exist finite subsets $I \subset A$, $J \subset B$ such that if $I' \supset I$, $J' \supset J$ are finite subsets of $A$, $B$ then
$$\left\| \sum_{\alpha \in I'} x_\alpha - x \right\| < \epsilon, \qquad \left\| \sum_{\beta \in J'} y_\beta - y \right\| < \epsilon.$$
Since
$$\begin{aligned} \sum_{(\alpha,\beta) \in I' \times J'} \langle x_\alpha, y_\beta \rangle - \langle x, y \rangle &= \left\langle \sum_{\alpha \in I'} x_\alpha, \sum_{\beta \in J'} y_\beta \right\rangle - \langle x, y \rangle \\ &= \left\langle x, \sum_{\beta \in J'} y_\beta - y \right\rangle + \left\langle \sum_{\alpha \in I'} x_\alpha - x, y \right\rangle \\ &\quad + \left\langle \sum_{\alpha \in I'} x_\alpha - x, \sum_{\beta \in J'} y_\beta - y \right\rangle, \end{aligned}$$

it follows from the Cauchy-Schwarz inequality that

$$\left|\sum_{(\alpha,\beta)\in I'\times J'} \langle x_\alpha, y_\beta\rangle - \langle x, y\rangle\right| < \epsilon\|x\| + \epsilon\|y\| + \epsilon^2.$$

- Therefore, given any $\epsilon > 0$, we can find a finite set $I \times J \subset A \times B$ such that if $I' \times J' \supset I \times J$ is any finite rectangle in $A \times B$, then

$$\left|\sum_{(\alpha,\beta)\in I'\times J'} \langle x_\alpha, y_\beta\rangle - \langle x, y\rangle\right| < \epsilon.$$

- Unfortunately, this is not quite sufficient to prove that the unordered sum

$$\sum_{(\alpha,\beta)\in A\times B} \langle x_\alpha, y_\beta\rangle$$

converges because we have to prove this inequality for every finite subset $K \supset I \times J$ of $A \times B$, not just rectangles of the form $K = I' \times J'$.

- The difficulty here is that if $K \subset I' \times J'$ we cannot say that

$$\left|\sum_{(\alpha,\beta)\in K} \langle x_\alpha, y_\beta\rangle - \langle x, y\rangle\right| \le \left|\sum_{(\alpha,\beta)\in I'\times J'} \langle x_\alpha, y_\beta\rangle - \langle x, y\rangle\right|.$$

We would be able to say this, however, if each term in the sum were a nonnegative real number. Thus, it appears necessary to split $\langle x_\alpha, y_\beta\rangle$ into its real and imaginary parts, and prove convergence of the unordered sums of the positive and negative terms in each sum, which would imply convergence of the original complex unordered sum. We omit a detailed proof. (Let me know if you have a simpler one!)

**Problem 3.** Let $\{e_\alpha \mid \alpha \in A\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Define

$$\mathcal{M} = \left\{ \sum_{\alpha\in A} c_\alpha e_\alpha \mid c_\alpha \in \mathbb{C},\ \sum_{\alpha\in A} |c_\alpha|^2 < \infty \right\}.$$

Prove that $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$.

**Solution.**

- The set $\mathcal{M}$ is well-defined since $\sum_{\alpha\in A} c_\alpha e_\alpha$ converges if and only if $\sum_{\alpha\in A} |c_\alpha|^2$ converges.

- Note that convergent unordered sums can be added term-by-term. Proof: Suppose that

$$\sum_{\alpha\in A} x_\alpha = x, \qquad \sum_{\alpha\in A} y_\alpha = y.$$

Given $\epsilon > 0$, there are finite sets $I \subset A$, $J \subset A$ such that if $I' \supset I$, $J' \supset J$ are finite subsets of $A$, then

$$\left\| \sum_{\alpha\in I'} x_\alpha - x \right\| < \frac{\epsilon}{2}, \qquad \left\| \sum_{\alpha\in J'} y_\alpha - y \right\| < \frac{\epsilon}{2}.$$

It follows that if $I \cup J \subset K$ is a finite subset of $A$, then

$$\begin{aligned}\left\| \sum_{\alpha\in K} (x_\alpha + y_\alpha) - (x+y) \right\| &\leq \left\| \sum_{\alpha\in K} x_\alpha - x \right\| + \left\| \sum_{\alpha\in K} y_\alpha - y \right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

- If $x, y \in \mathcal{M}$, with

$$x = \sum_{\alpha\in A} a_\alpha e_\alpha, \qquad y = \sum_{\alpha} b_\alpha e_\alpha$$

then

$$x + y = \sum_{\alpha\in A} (a_\alpha + b_\alpha) e_\alpha.$$

Since

$$|a_\alpha + b_\alpha|^2 \le (|a_\alpha| + |b_\alpha|)^2 \le (2\max\{|a_\alpha|, |b_\alpha|\})^2 \le 4\left(|a_\alpha|^2 + |b_\alpha|^2\right),$$

it follows that

$$\sum_{\alpha\in A} |a_\alpha + b_\alpha|^2 \le 4\left(\sum_{\alpha\in A} |a_\alpha|^2 + \sum_{\alpha\in A} |b_\alpha|^2\right) < \infty,$$

so $x + y \in \mathcal{M}$.

- Similarly, we also have $\lambda x \in \mathcal{M}$ for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{M}$, so $\mathcal{M}$ is a linear space.
- To prove that $\mathcal{M}$ is closed, first note that $\overline{\mathcal{M}} \cap \mathcal{M}^\perp = \{0\}$. This follows from the projection theorem, but it is easy to show directly: If $x \in \overline{\mathcal{M}} \cap \mathcal{M}^\perp$, then since $\mathcal{M}^\perp = \overline{\mathcal{M}}^\perp$ we have $x \perp x$, so $x = 0$.
- Suppose that $x \in \overline{\mathcal{M}}$. Let

  $$y = \sum_{\alpha\in A} \langle e_\alpha, x\rangle e_\alpha.$$

  According to Bessel's inequality,

  $$\sum_{\alpha\in A} |\langle e_\alpha, x\rangle|^2 \le \|x\|^2,$$

  so $y \in \mathcal{M}$. Moreover, $\langle e_\alpha, x\rangle = \langle e_\alpha, y\rangle$ for all $\alpha \in A$, which implies that $x - y \in \mathcal{M}^\perp$. Since $x, y \in \overline{\mathcal{M}}$, it follows that $x - y \in \overline{\mathcal{M}} \cap \mathcal{M}^\perp$, so $x = y$ and $x \in \mathcal{M}$, which implies that $\mathcal{M}$ is closed.