Solutions: Problem Set 3
Math 201B, Winter 2007

Problem 1. Prove that an infinite-dimensional Hilbert space is a separable metric space if and only if it has a countable orthonormal basis.

Solution.

- If $\mathcal{H}$ is a finite-dimensional Hilbert space with orthonormal basis

$$\{e_n \mid 1 \le n \le d\},$$

then

$$D = \left\{ \sum_{n=1}^{d} c_n e_n \mid c_n = q_n + i r_n \text{ with } q_n, r_n \in \mathbb{Q} \right\}$$

is a countable dense subset of $\mathcal{H}$.

- If $\mathcal{H}$ is an infinite-dimensional Hilbert space with countable orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$, then

$$D = \left\{ \sum_{n=1}^{N} c_n e_n \mid N \in \mathbb{N},\ c_n = q_n + i r_n \text{ with } q_n, r_n \in \mathbb{Q} \right\}$$

is a countable dense subset. Thus, $\mathcal{H}$ is separable if it has a countable orthonormal basis.

- Suppose that $\mathcal{H}$ has an uncountable orthonormal basis,

$$E = \{e_\alpha \mid \alpha \in A\},$$

and let D be a dense subset of $\mathcal{H}$.

- The orthonormality of E implies that if $\alpha \neq \beta$, then

$$\|e_\alpha - e_\beta\|^2 = \|e_\alpha\|^2 + \|e_\beta\|^2 = 2.$$

The open balls $B_{\sqrt{2}/2}(e_\alpha)$ are therefore disjoint, and, since D is dense, each ball contains at least one point $x_\alpha \in D$, say. The map $\alpha \mapsto x_\alpha$ is a one-to-one map of A into D, so the cardinality of D is greater than or equal to the cardinality of A. It follows that no dense subset of $\mathcal{H}$ is countable, so $\mathcal{H}$ is not separable.

Problem 2. Prove that if M is a dense linear subspace of a separable Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis consisting of elements in M.

Solution.

- If $\mathcal{H}$ is finite-dimensional, then every linear subspace is closed. Thus, the only dense linear subspace of $\mathcal{H}$ is $\mathcal{H}$ itself, and the result follows from the fact that $\mathcal{H}$ has an orthonormal basis.

- Suppose that $\mathcal{H}$ is infinite-dimensional. Since $\mathcal{H}$ is separable, it has a countable dense subset $\{x_n \mid n \in \mathbb{N}\}$, which need not be a subset of M. Since M is dense in $\mathcal{H}$, for each $n \in \mathbb{N}$, there exists a sequence (x_{mn}) in M such that $x_{mn} \to x_n$ as $m \to \infty$. The set $\{x_{mn} \mid m, n \in \mathbb{N}\}$ is then a countable subset of M that is dense in $\mathcal{H}$.

- Let D be a subset of M that is dense in $\mathcal{H}$, and let $B = \{x_n \mid n \in \mathbb{N}\}$ be a maximal linearly independent subset of D. Then the linear span of B, meaning all finite linear combinations of elements of B, contains D so it is dense in $\mathcal{H}$. The closed linear span of B is therefore equal to $\mathcal{H}$.

- Gram-Schmidt orthonormalization of B gives an orthonormal set

  $$E = \{e_n \mid n \in \mathbb{N}\}$$

  whose closed linear span is equal to that of B, meaning that E is an orthonormal basis of $\mathcal{H}$. Moreover, since each $x_n \in M$ and each $e_n \in E$ is a finite linear combination of $\{x_1, \dots, x_n\}$, it follows that $e_n \in M$. Thus, E is an orthonormal basis of $\mathcal{H}$ consisting of elements of M.

Problem 3. Define the Legendre polynomials P_n by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

(a) Compute the first four Legendre polynomials, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$.
(b) Show that the Legendre polynomials are orthogonal in $L^2([-1,1])$.
(c) Show that the Legendre polynomials are obtained by Gram-Schmidt orthogonalization of the monomials $\{1, x, x^2, \ldots\}$ in $L^2([-1,1])$.
(d) Show that

$$\int_{-1}^{1} P_n(x)^2\,dx = \frac{2}{2n+1}.$$

(e) Show that the Legendre polynomial P_n is an eigenfunction of the differential operator

$$L = -\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}$$

with eigenvalue $\lambda_n = n(n+1)$, meaning that

$$LP_n = \lambda_n P_n.$$

(f) Compute the polynomial $q(x)$ of degree 2 that is 'closest' to e^x on $[-1,1]$, in the sense that

$$\int_{-1}^{1} \left|e^x - q(x)\right|^2\,dx = \min\left\{\int_{-1}^{1}\left|e^x - f(x)\right|^2\,dx \mid f(x) = ax^2+bx+c\right\}.$$

Solution.

- (a) The first few Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \qquad P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}.$$

- To prove that P_m is orthogonal to P_n, where we may assume $m < n$ without loss of generality, it suffices to prove that x^m is orthogonal to P_n for every $m < n$. It then follows by linearity that every polynomial of degree $m < n$ is orthogonal to P_n, including, in particular, P_m.

- Integrating by parts m-times, we compute that

$$\begin{aligned}
\langle x^m, P_n\rangle &= \frac{1}{2^n n!}\int_{-1}^{1} x^m \frac{d^n}{dx^n}\left(x^2-1\right)^n\, dx \\
&= \frac{(-1)^m}{2^n n!}\int_{-1}^{1} \frac{d^m}{dx^m}\left(x^m\right) \frac{d^{n-m}}{dx^{n-m}}\left(x^2-1\right)^n\, dx \\
&= \frac{(-1)^m m!}{2^n n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}\left(x^2-1\right)^n\right]_{-1}^{1} \\
&= 0.
\end{aligned}$$

  All of the boundary terms vanish at $x = \pm 1$ because, for $1 \leq k \leq n$, the polynomial

$$\frac{d^{n-k}}{dx^{n-k}}\left(x^2-1\right)^n$$

  has a factor $(x^2-1)^k$. Hence $x^m \perp P_n$ for $m < n$, and $P_m \perp P_n$ for $m \neq n$.

- (c) The linear subspace of polynomials of degree n has dimension $n+1$. The orthogonal complement of the polynomials of degree $n-1$ in the space of polynomials of degree n is equal to 1, and therefore $\{P_n\}$ is a basis of the orthogonal complement. The Gram-Schmidt orthogonalization of the monomials gives a polynomial of degree n in this complement, so it gives the Legendre polynomials up to normalization.

- (d) Integrating by parts as in (a), we compute that

$$\begin{aligned}
\langle P_n, P_n\rangle &= \frac{1}{\left(2^n n!\right)^2}\int_{-1}^{1} \frac{d^n}{dx^n}\left(x^2-1\right)^n \frac{d^n}{dx^n}\left(x^2-1\right)^n\, dx \\
&= \frac{(-1)^n}{\left(2^n n!\right)^2}\int_{-1}^{1} \left(x^2-1\right)^n \frac{d^{2n}}{dx^{2n}}\left(x^2-1\right)^n\, dx \\
&= \frac{(-1)^n (2n)!}{\left(2^n n!\right)^2}\int_{-1}^{1} \left(x^2-1\right)^n\, dx.
\end{aligned}$$

  Using integration by parts again, we get

$$\int_{-1}^{1}\left(x^2-1\right)^n\, dx = \int_{-1}^{1}(x-1)^n (x+1)^n\, dx$$

$$
\begin{aligned}
&= -\frac{n}{n+1}\int_{-1}^{1}(x-1)^{n-1}(x+1)^{n+1}\,dx \\
&= \frac{(-1)^n n(n-1)\dots 1}{(n+1)(n+2)\dots(2n)}\int_{-1}^{1}(x+1)^{2n}\,dx \\
&= \frac{(n!)^2 2^{2n+1}}{(2n)!(2n+1)}.
\end{aligned}
$$

Using this integral in the expression for the inner product of P_n, and simplifying the result we get

$$
\|P_n\|^2 = \langle P_n, P_n\rangle = \frac{2}{2n+1}
$$

- (e) We write $D = d/dx$. Leibnitz's rule for the nth derivative of a product gives

$$
D^n(fg) = \sum_{k=0}^{n}\binom{n}{k} D^k f \cdot D^{n-k} g.
$$

In particular, since $D^k x^m = 0$ for $k > m$,

$$
\begin{aligned}
D^n(xf) &= xD^n f + nD^{n-1}f, \\
D^n\left(x^2 f\right) &= x^2 D^n f + 2nxD^{n-1}f + n(n-1)D^{n-2}f.
\end{aligned}
$$

- Let $u(x) = (x^2-1)^n$. Then

$$
\left(x^2-1\right)Du = 2nxu.
$$

We apply D^{n+1} to this equation, and use Leibnitz's rule to expand the derivatives, which gives

$$
\begin{aligned}
&\left(x^2-1\right)D^{n+2}u + (n+1)\cdot 2xD^{n+1}u + (n+1)nD^n u \\
&\qquad = 2nxD^{n+1}u + 2n(n+1)D^n u.
\end{aligned}
$$

After simplification we obtain that

$$
\left(x^2-1\right)D^{n+2}u + 2xD^{n+1}u - n(n+1)D^n u = 0,
$$

which implies that

$$
\left(x^2-1\right)D^2 P_n + 2xDP_n - n(n+1)P_n = 0.
$$

This equation is equivalent to $LP_n = \lambda_n P_n$.

- (f) By the projection theorem, the closest polynomial of degree N to $f \in L^2([-1,1])$ is the one such that the error $f - q$ is orthogonal to the linear space of polynomials of degree N, meaning that

$$q = \sum_{n=1}^{N} c_n P_n$$

where the $\{c_n \mid n = 0, 1, 2, \ldots\}$ are the Fourier coefficients of f with respect to the Legendre polynomials $\{P_n \mid n = 0, 1, 2, \ldots\}$,

$$c_n = \frac{\langle P_n, f \rangle}{\|P_n\|^2}.$$

- If $f(x) = e^x$, then we compute that

$$\begin{aligned}
\langle P_0, f \rangle &= \int_{-1}^{1} 1 \cdot e^x \, dx \\
&= [e^x]_{-1}^{1} \\
&= e - \frac{1}{e}, \\
\langle P_1, f \rangle &= \int_{-1}^{1} x e^x \, dx \\
&= [x e^x - e^x]_{-1}^{1} \\
&= \frac{2}{e}, \\
\langle P_2, f \rangle &= \int_{-1}^{1} \left(\frac{3}{2} x^2 - \frac{1}{2} \right) e^x \, dx \\
&= \left[\frac{3}{2} x^2 e^x - 3 x e^x + 3 e^x - \frac{1}{2} e^x \right]_{-1}^{1} \\
&= e - \frac{7}{e}.
\end{aligned}$$

Using the normalization of the Legendre polynomials from (d), we find that the closest quadratic polynomial q to e^x in $L^2([-1,1])$ is

$$\begin{aligned}
q(x) &= \frac{1}{2}\left(e - \frac{1}{e}\right) + \frac{3}{2}\left(\frac{2}{e}\right) x + \frac{5}{2}\left(e - \frac{7}{e}\right)\left(\frac{3}{2} x^2 - \frac{1}{2}\right) \\
&= -\frac{3}{4}\left(e - \frac{11}{e}\right) + \frac{3}{e} x + \frac{15}{4}\left(e - \frac{7}{e}\right) x^2.
\end{aligned}$$

Problem 4. Define the Hermite polynomials H_n by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

(a) Define

$$\phi_n(x) = e^{-x^2/2} H_n(x).$$

Show that $\{\phi_n \mid n = 0, 1, 2, \ldots\}$ is an orthogonal set in $L^2(\mathbb{R})$.

(b) Show that the nth Hermite function ϕ_n is an eigenfunction of the linear operator

$$H = -\frac{d^2}{dx^2} + x^2$$

with eigenvalue

$$\lambda_n = 2n + 1.$$

Solution.

- (a) It is sufficient to show that ϕ_n is orthogonal to $e^{-x^2/2}x^m$ for each $m < n$, since then ϕ_n is orthogonal to every function of the form $e^{-x^2/2}p_m$, where p_m is a polynomial of degree $m < n$, and hence in particular to ϕ_m.

- Integrating by parts m-times, and using the fact that $p\, e^{-x^2/2} \to 0$ as $|x| \to \infty$ for every polynomial p, we compute that

$$\begin{aligned}
\left\langle e^{-x^2/2}x^m, \phi_n \right\rangle &= (-1)^n \int_{-\infty}^{\infty} x^m \frac{d^n}{dx^n}\left(e^{-x^2}\right) dx \\
&= (-1)^{m+n} m! \int_{-\infty}^{\infty} \frac{d^{n-m}}{dx^{n-m}}\left(e^{-x^2}\right) dx \\
&= (-1)^{m+n} m! \left[\frac{d^{n-m-1}}{dx^{n-m-1}}\left(e^{-x^2}\right)\right]_{-\infty}^{\infty} \\
&= 0,
\end{aligned}$$

  which proves the result.

- (b) Let

$$A = \frac{d}{dx} + x, \qquad A^* = -\frac{d}{dx} + x.$$

We show below that

$$\frac{dH_n}{dx} = 2nH_{n-1} = -H_{n+1} + 2xH_n. \tag{1}$$

- Using this result, we compute that

$$\begin{aligned} A\phi_n &= \left(\frac{d}{dx} + x\right)\left(e^{-x^2/2}H_n\right) \\ &= e^{-x^2/2}\frac{dH_n}{dx} \\ &= 2ne^{-x^2/2}H_{n-1} \\ &= 2n\phi_{n-1}, \end{aligned}$$

and

$$\begin{aligned} A^*\phi_n &= \left(-\frac{d}{dx} + x\right)\left(e^{-x^2/2}H_n\right) \\ &= e^{-x^2/2}\left(-\frac{dH_n}{dx} + 2xH_n\right) \\ &= e^{-x^2/2}H_{n+1} \\ &= \phi_{n+1}, \end{aligned}$$

- The product rule $(xf)' = xf' + f$ implies that

$$\frac{d}{dx}x = x\frac{d}{dx} + 1.$$

Hence

$$\begin{aligned} AA^* &= \left(\frac{d}{dx} + x\right)\left(-\frac{d}{dx} + x\right) \\ &= -\frac{d^2}{dx^2} + \frac{d}{dx}x - x\frac{d}{dx} + x^2 \\ &= H + 1, \end{aligned}$$

so $H = AA^* - 1$.

- It follows that

$$\begin{aligned}
H\phi_n &= (AA^* - 1)\,\phi_n \\
&= AA^*\phi_n - \phi_n \\
&= A\phi_{n+1} - \phi_n \\
&= 2(n+1)\phi_n - \phi_n \\
&= (2n+1)\,\phi_n.
\end{aligned}$$

- Finally, we prove (1). First, using the product rule, we get

$$\begin{aligned}
\frac{dH_n}{dx} &= (-1)^n \frac{d}{dx}\left[e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)\right] \\
&= (-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + (-1)^n 2xe^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) \\
&= -H_{n+1} + 2xH_n. \qquad (2)
\end{aligned}$$

- Second, carrying out one differentiation and using the Leibnitz formula for the nth derivative of a product, we get

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) &= \frac{d^n}{dx^n}\left(-2xe^{-x^2}\right) \\
&= -2x\frac{d^n}{dx^n}\left(e^{-x^2}\right) - 2n\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x^2}\right).
\end{aligned}$$

  Multiplying this equation by $(-1)^{n+1}e^{x^2}$ and using the definition of the Hermite polynomials, we get the recurrence relation

$$H_{n+1} = 2xH_n - 2nH_{n-1}.$$

  Using this equation to eliminate H_{n+1} from (2), we find that

$$\frac{dH_n}{dx} = 2nH_{n-1}.$$

Remark. It follows from the Weierstrass approximation theorem that both the Legendre polynomials and the Hermite functions are complete orthonormal sets, and hence they provide orthonormal bases of $L^2([-1,1])$ and $L^2(\mathbb{R})$, respectively.