Solutions: Problem Set 4
Math 201B, Winter 2007

Problem 1. (a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Use the monotone convergence theorem to show that $f \in L^1(\mathbb{R})$.
(b) Suppose that $\{r_n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ is an enumeration of the rational numbers. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n),$$

where f is the function defined in (a). Show that $g \in L^1(\mathbb{R})$, even though it is unbounded on every interval.

Solution.

- Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} x^{-1/2} & \text{if } 1/n < x < 1 - 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

  Then (f_n) is a monotone increasing sequence of nonnegative, measurable functions (since $f^{-1}((a, \infty))$ is open and therefore measurable for every $a \in \mathbb{R}$) which converges pointwise to f on $\mathbb{R}$ (so f is measurable).

- Each f_n is Riemann integrable on $[0, 1]$, and

$$\begin{aligned} \int_{\mathbb{R}} f_n \, dx &= \int_{1/n}^{1-1/n} x^{-1/2} \, dx \\ &= \left[2x^{1/2}\right]_{1/n}^{1-1/n} \\ &\to 2 \qquad \text{as } n \to \infty. \end{aligned}$$

- The monotone convergence theorem implies that

$$\int_{\mathbb{R}} f \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dx = 2 < \infty,$$

  so $f \in L^1(\mathbb{R})$.

- (b) Let

$$g_N(x) = \sum_{n=1}^{N} \frac{1}{2^n} f(x - r_n).$$

  Then (g_N) is a monotone increasing sequence, since $f \geq 0$, that converges pointwise to g. By the linearity of the integral and the translation invariance of Lebesgue measure,

$$\begin{aligned} \int_{\mathbb{R}} g_N \, dx &= \sum_{n=1}^{N} \frac{1}{2^n} \int_{\mathbb{R}} f(x - r_n) \, dx \\ &= 2 \sum_{n=1}^{N} \frac{1}{2^n} \\ &\to 2 \qquad \text{as } N \to \infty. \end{aligned}$$

  Hence, by the monotone convergence theorem

$$\int_{\mathbb{R}} g \, dx = 2,$$

  so g is integrable. (In particular, the series defining g diverges to ∞ on at most a set of measure zero.)

- The function g is unbounded in any neighborhood of $r_n \in \mathbb{Q}$, and therefore on any open interval since the rationals are dense in $\mathbb{R}$.

Remark. This example illustrates that integrable functions are not necessarily as 'nice' as one might imagine; in particular they do not necessarily approach 0 at infinity.

Problem 2. If $f \in L^1(\mathbb{R})$, prove that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f\, dx = 0.$$

Give an example to show that this result need not be true if f is not integrable on $\mathbb{R}$.

Solution.

- Let
$$f_n = \frac{1}{2n} \chi_{[-n,n]} f,$$
where $\chi_{[-n,n]}$ is the characteristic function of the interval $[-n, n]$. Then
$$\int f_n\, dx = \frac{1}{2n} \int_{-n}^{n} f\, dx.$$

- We have $f_n(x) \to 0$ as $n \to \infty$ whenever $f(x) \neq \pm\infty$, so $f_n \to 0$ pointwise a.e. on $\mathbb{R}$. Also, for $n \geq 1$,
$$|f_n| \leq \frac{1}{2}|f| \in L^1(\mathbb{R}).$$

- The Lebesgue dominated convergence theorem implies that
$$\lim_{n\to\infty} \int f_n\, dx = \int \lim_{n\to\infty} f_n\, dx = \int 0\, dx = 0,$$
which proves the result

- If $f = 1$, then
$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} f\, dx = 1.$$
In this case the sequence
$$f_n = \frac{1}{2n} \chi_{[-n,n]}$$
converges pointwise (and even uniformly) to 0 on $\mathbb{R}$ as $n \to \infty$, but the integrals do not. Note that the convergence is not monotone and the sequence (f_n) is not dominated by any integrable function.

Problem 3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1/x^2 & \text{if } 0 < y < x < 1, \\ -1/y^2 & \text{if } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the following integrals:

$$\int_{\mathbb{R}^2} |f(x,y)| \, dx dy; \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) \, dx \right) dy; \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) \, dy \right) dx.$$

Are your results consistent with Fubini's theorem?

Solution.

- If $y \leq 0$ or $y \geq 1$, then $f(x,y) = 0$, and

$$\int_{\mathbb{R}} f(x,y) \, dx = 0.$$

If $0 < y < 1$, then

$$\begin{aligned} \int_{\mathbb{R}} f(x,y) \, dx &= \int_0^y -\frac{1}{y^2} \, dx + \int_y^1 \frac{1}{x^2} \, dx \\ &= \left[-\frac{x}{y^2} \right]_{x=0}^{x=y} + \left[-\frac{1}{x} \right]_{x=y}^{x=1} \\ &= -\frac{1}{y} - 1 + \frac{1}{y} \\ &= -1. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) \, dx \right) dy = \int_0^1 -1 \, dy = -1.$$

- If $x \leq 0$ or $x \geq 1$, then $f(x,y) = 0$, and

$$\int_{\mathbb{R}} f(x,y) \, dy = 0.$$

If $0 < x < 1$, then

$$\begin{aligned}
\int_{\mathbb{R}} f(x,y)\,dy &= \int_0^x \frac{1}{x^2}\,dy + \int_x^1 -\frac{1}{y^2}\,dy \\
&= \left[\frac{y}{x^2}\right]_{y=0}^{y=x} + \left[\frac{1}{y}\right]_{y=x}^{y=1} \\
&= \frac{1}{x} + 1 - \frac{1}{x} \\
&= 1.
\end{aligned}$$

It follows that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dy \right) dx = \int_0^1 1\,dx = 1.$$

- According to Fubini's theorem, we can evaluate the integral of $|f| \geq 0$ as an iterated integral (in either order):

$$\begin{aligned}
\int_{\mathbb{R}^2} |f(x,y)|\,dxdy &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|\,dx \right) dy \\
&= \int_0^1 \left(\int_0^y \frac{1}{y^2}\,dx + \int_y^1 \frac{1}{x^2}\,dx \right) dy \\
&= \int_0^1 \left(\left[\frac{x}{y^2}\right]_{x=0}^{x=y} + \left[-\frac{1}{x}\right]_{x=y}^{x=1} \right) dy \\
&= \int_0^1 \left(\frac{2}{y} - 1 \right) dy \\
&= \lim_{n\to\infty} \int_{1/n}^1 \left(\frac{2}{y} - 1 \right) dy \\
&= \lim_{n\to\infty} \left[2\log y - y\right]_{1/n}^1 \\
&= \infty.
\end{aligned}$$

- This example shows that if one drops the assumption that $f \in L^1$ in Fubini's theorem then the iterated integrals with different orders need not be equal. Also note that both $\int f_+\,dxdy$ and $\int f_-\,dxdy$ are equal to ∞, so $\int f\,dxdy$ is undefined.

Problem 4. Define $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$f(x, y) = xe^{-x^2\left(1+y^2\right)}.$$

Compute the iterated integrals with respect to x, y and y, x, and use Fubini's theorem to show that

$$\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

Solution.

- Integrating with respect to x followed by y, and using the monotone convergence theorem, we get

$$\begin{aligned}
\int_0^\infty \left(\int_0^\infty f(x, y)\, dx \right) dy &= \int_0^\infty \left(\lim_{n\to\infty} \int_0^n xe^{-x^2\left(1+y^2\right)}\, dx \right) dy \\
&= \frac{1}{2} \int_0^\infty \left(\lim_{n\to\infty} \left[-\frac{e^{-x^2\left(1+y^2\right)}}{1+y^2} \right]_{x=0}^{x=n} \right) dy \\
&= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2}\, dy \\
&= \frac{1}{2} \lim_{n\to\infty} \int_0^n \frac{1}{1+y^2}\, dy \\
&= \frac{1}{2} \lim_{n\to\infty} \left[\tan^{-1} y \right]_0^n \\
&= \frac{\pi}{4}.
\end{aligned}$$

- Integrating with respect to y followed by x, using the monotone convergence theorem, and making the change of variables $t = xy$, we get

$$\begin{aligned}
\int_0^\infty \left(\int_0^\infty f(x, y)\, dy \right) dx &= \int_0^\infty \left(\lim_{n\to\infty} \int_0^n xe^{-x^2\left(1+y^2\right)}\, dy \right) dx \\
&= \int_0^\infty \left(\lim_{n\to\infty} \int_0^{nx} e^{-\left(x^2+t^2\right)}\, dt \right) dx \\
&= \left(\int_0^\infty e^{-x^2}\, dx \right) \left(\int_0^\infty e^{-t^2}\, dt \right) \\
&= \left(\int_0^\infty e^{-t^2}\, dt \right)^2.
\end{aligned}$$

- The function f is non-negative, so Fubini's theorem implies that the iterated integrals of f are equal, and both are finite if one is finite. Equating the two iterated integrals and taking the square-root, we get the result.

Problem 5. In a normed space X, let

$$B_r(a) = \{x \in X \mid \|x - a\| < r\}.$$

be the ball of radius $r > 0$ centered at $a \in X$. Lebesgue measure m on $\mathbb{R}^d$ (with the Euclidean norm, say) has the properties that every ball is measurable with finite, nonzero measure, and the measure of a ball is invariant under translations. That is, for every $0 < r < \infty$ and $a \in X$

$$0 < m(B_r(a)) < \infty, \qquad m(B_r(a)) = m(B_r(0)).$$

Prove that it is not possible to define a measure with these properties on an infinite-dimensional Hilbert space.

Solution.

- Since $\mathcal{H}$ is an infinite-dimensional Hilbert space, it contains a countably infinite orthonormal set $\{e_n \mid n \in \mathbb{N}\}$.

- If $k \neq n$ then the Pythagorean theorem implies that

$$\|e_k - e_n\| = \sqrt{\|e_k\|^2 + \|e_n\|^2} = \sqrt{2}.$$

  It follows that the balls $B_{\sqrt{2}/2}(e_k)$ and $B_{\sqrt{2}/2}(e_n)$ are disjoint.

- Suppose that

$$m\left(B_{\sqrt{2}/2}(0)\right) = \epsilon.$$

  Then, by translation invariance,

$$m\left(B_{\sqrt{2}/2}(e_n)\right) = \epsilon \qquad \text{for every } n \in \mathbb{N}.$$

- Since the balls are disjoint, the countable additivity of m implies that if $\epsilon > 0$

$$\begin{aligned} m\left(\bigcup_{n\in\mathbb{N}} B_{\sqrt{2}/2}(e_n)\right) &= \sum_{n\in\mathbb{N}} m\left(B_{\sqrt{2}/2}(e_n)\right) \\ &= \sum_{n\in\mathbb{N}} \epsilon \\ &= \infty. \end{aligned}$$

- If $x \in B_{\sqrt{2}/2}(e_n)$, then

$$\|x\| \leq \|e_n\| + \|x - e_n\| < 1 + \frac{\sqrt{2}}{2}.$$

It follows that

$$\bigcup_{n \in \mathbb{N}} B_{\sqrt{2}/2}(e_n) \subset B_{1+\sqrt{2}/2}(0).$$

- The additivity and non-negativity of m implies that

$$m\left(\bigcup_{n \in \mathbb{N}} B_{\sqrt{2}/2}(e_n)\right) \leq m\left(B_{1+\sqrt{2}/2}(0)\right),$$

so if $m\left(B_{\sqrt{2}/2}(0)\right) > 0$ then

$$m\left(B_{1+\sqrt{2}/2}(0)\right) = \infty.$$

- Thus, if the measure of the smaller ball, with radius $\sqrt{2}/2$, is nonzero, the measure of the larger ball, with radius $(1 + \sqrt{2}/2)$, is infinite, so there is no translation invariant measure that assigns a non-zero, finite measure to every ball.