Solutions: Problem Set 5
Math 201B, Winter 2007

Problem 1. Define $f : \mathbb{T} \to \mathbb{R}$ by

$$f(x) = x^2 \qquad \text{for } -\pi \le x \le \pi.$$

(a) Compute the Fourier coefficients of f.
(b) Use Parseval's theorem to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution.

- (a) If $n \neq 0$, we have

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-in} \left(x^2 - \frac{2}{-in} x + \frac{2}{(-in)^2} \right) e^{-inx} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\sqrt{2\pi}} \frac{4\pi(-1)^n}{n^2}.
\end{aligned}$$

  If $n = 0$, we have

$$\begin{aligned}
\hat{f}(0) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2\, dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{2\pi^3}{3}.
\end{aligned}$$

- (b) By Parseval's theorem,

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right|^2 = \int_{\mathbb{T}} |f(x)|^2\, dx.$$

  We compute that

$$\int_{\mathbb{T}} |f(x)|^2\, dx = \int_{-\pi}^{\pi} x^4\, dx = \frac{2\pi^5}{5}.$$

Using (a), we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2 &= \frac{1}{2\pi}\frac{4\pi^6}{9} + 2\cdot\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{16\pi^2}{n^4} \\
&= \frac{2\pi^5}{9} + 16\pi\sum_{n=1}^{\infty}\frac{1}{n^4}.
\end{aligned}$$

Equating these two expressions and solving for $\sum_{n=1}^{\infty} 1/n^4$, we get the result.

Problem 2. Suppose that $(\phi_n)_{n=1}^{\infty}$ is an approximate identity on $\mathbb{T}$ and $f \in L^1(\mathbb{T})$.
(a) Prove that for every $n \in \mathbb{N}$

$$\|\phi_n * f\|_1 \leq \|f\|_1.$$

(b) Prove that

$$\|\phi_n * f - f\|_1 \to 0 \qquad \text{as } n \to \infty.$$

(c) If $f, g \in L^1(\mathbb{T})$ have the same Fourier coefficients, prove that $f = g$.

Solution.

- (a) Using Fubini's theorem for nonnegative functions and the fact that an approximate identity is nonnegative with integral equal to 1, we get

$$\begin{aligned}
\|\phi_n * f\|_1 &= \int_{\mathbb{T}} |\phi_n * f(x)| \, dx \\
&= \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \phi_n(x-y) f(y) \, dy \right| \, dx \\
&\leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \phi_n(x-y) |f(y)| \, dy \right) \, dx \\
&= \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \phi_n(x-y) |f(y)| \, dx \right) \, dy \\
&= \int_{\mathbb{T}} |f(y)| \, dy \\
&= \|f\|_1.
\end{aligned}$$

- (b) Note that if $g \in C(\mathbb{T})$, then

$$\|g\|_1 = \int_{\mathbb{T}} |g(x)| \, dx \leq 2\pi \|g\|_\infty.$$

Let $\epsilon > 0$. Since the continuous functions are dense in $L^1(\mathbb{T})$, there exists $g \in C(\mathbb{T})$ such that $\|f - g\|_1 < \epsilon/3$. Since ϕ_n is an approximate identity, $\phi_n * g \to g$ uniformly as $n \to \infty$, and there is $N \in \mathbb{N}$ such that

$$\|\phi_n * g - g\|_\infty < \frac{\epsilon}{6\pi} \qquad \text{for all } n > N.$$

If $n > N$, then, using the result from (a), we get

$$\begin{aligned}\|\phi_n * f - f\|_1 &\le \|\phi_n * f - \phi_n * g\|_1 + \|\phi_n * g - g\|_1 + \|g - f\|_1 \\ &= \|\phi_n * (f - g)\|_1 + \|\phi_n * g - g\|_1 + \|g - f\|_1 \\ &\le \|f - g\|_1 + 2\pi \|\phi_n * g - g\|_\infty + \|g - f\|_1 \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\end{aligned}$$

which proves that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$.

- (c) If $f, g \in L^1(\mathbb{T})$ have the same Fourier coefficients, then the Fourier coefficients of $f - g$ are equal to 0. It therefore suffices to prove that if the Fourier coefficients of $f \in L^1(\mathbb{T})$ are equal to 0, then $f = 0$.

- If

  $$p(x) = \sum_{|n| \le N} c_n e^{inx}$$

  is a trigonometric polynomial, then

  $$(p * f)(x) = \sqrt{2\pi} \sum_{|n| \le N} c_n \hat{f}(n) e^{inx}.$$

  Thus, $p * f = 0$ if $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$.

- Let $(\phi_n)_{n=1}^\infty$ be an approximate identity consisting of trigonometric polynomials ϕ_n. Then $\phi_n * f = 0$ for every $n \in \mathbb{N}$ and, from (b), $\phi_n * f \to f$ in $L^1(\mathbb{T})$, so $f = 0$. This proves the result.

Remark. If $f, g \in L^2(\mathbb{T})$ have the same Fourier coefficients, then it is immediate that $f = g$ because the Fourier series converge to the functions in $L^2(\mathbb{T})$. This argument does not work for $f, g \in L^1(\mathbb{T})$ since, in general, the Fourier series of the functions need not converge in $L^1(\mathbb{T})$.

Problem 3. Consider the differential equation

$$-u'' + u = f.$$

(a) If $f \in L^2(\mathbb{T})$, use Fourier series to show that there is unique solution $u \in H^2(\mathbb{T})$.
(b) Show that $u = G * f$ for a suitable function G (called the Green's function).
(c) Show that $G \in H^s(\mathbb{T})$ for $s < 3/2$.

Solution.

- (a) Computing Fourier coefficients, we find that $u \in L^2(\mathbb{T})$ is a solution of the equation if and only if

$$-(in)^2 \hat{u}_n + \hat{u}_n = \hat{f}_n,$$

or

$$\hat{u}_n = \frac{\hat{f}_n}{1+n^2}.$$

If $u \in L^2(\mathbb{T})$ is the function with these Fourier coefficients, then, since

$$\|u\|_{H^2}^2 = \sum_{n\in\mathbb{Z}} \left(1+n^2\right)^2 |\hat{u}_n|^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}_n\right|^2 = \|f\|_2^2,$$

we see that $u \in H^2(\mathbb{T})$, which proves the result. In fact, we have shown that the linear operator $(-d^2/dx^2 + 1)$ is an isometric isomorphism of $H^2(\mathbb{T})$ onto $L^2(\mathbb{T})$.

- (b) By the convolution theorem,

$$u = G * f$$

where

$$\hat{G}_n = \frac{1}{\sqrt{2\pi}} \frac{1}{1+n^2}.$$

It follows that

$$G(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} e^{inx},$$

where the Fourier series converges in $L^2(\mathbb{T})$. (Since the Fourier coefficients of G are summable, it also converges uniformly.)

- (c) The function $G \in H^s(\mathbb{T})$ if and only if

$$\sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^s \left|\hat{G}_n\right|^2 < \infty,$$

or

$$\sum_{n \in \mathbb{Z}} \frac{1}{\left(1 + n^2\right)^{2-s}} < \infty.$$

The terms in this series behave like $1/n^{2(2-s)}$ as $n \to \infty$, and by comparison with the series $\sum 1/n^p$, the series converges if and only if

$$2(2 - s) > 1,$$

or $s < 3/2$.

Problem 4. Suppose that $f : \mathbb{T} \to \mathbb{C}$ is continuous, with Fourier coefficients

$$\hat{f}_n = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx}\, dx.$$

Let f_N denote the mean of the first $(N+1)$ partial sums of the Fourier series of f, meaning that

$$f_N(x) = \frac{1}{N+1} \sum_{M=0}^{N} \left(\frac{1}{\sqrt{2\pi}} \sum_{m=-M}^{M} \hat{f}_m e^{imx} \right).$$

(a) Show that $f_N = K_N * f$ where $K_N : \mathbb{T} \to \mathbb{R}$ is the Fejér kernel, given by

$$K_N(x) = \frac{1}{2\pi} \frac{1}{N+1} \sum_{n=-N}^{N} \left(N + 1 - |n|\right) e^{inx}.$$

(b) Show that $K_N(x)$ may also be written as

$$\begin{aligned} K_N(x) &= \frac{1}{2\pi} \frac{1}{N+1} \left[\frac{\sin\left((N+1)\, x/2\right)}{\sin\left(x/2\right)} \right]^2 \qquad x \neq 0, \\ K_N(0) &= \frac{1}{2\pi}(N+1). \end{aligned}$$

(c) Show that K_N is an approximate identity. What can you say about the convergence of f_N to f as $N \to \infty$?

Solution.

- (a) Writing $\hat{f}_m$ in terms of f and rearranging the result, we have

$$\begin{aligned} f_N(x) &= \frac{1}{\sqrt{2\pi}} \frac{1}{N+1} \sum_{M=0}^{N} \sum_{m=-M}^{M} \hat{f}_m e^{imx} \\ &= \frac{1}{2\pi} \frac{1}{N+1} \sum_{M=0}^{N} \sum_{m=-M}^{M} \left(\int_{\mathbb{T}} f(y) e^{-imy}\, dy \right) e^{imx} \\ &= \int_{\mathbb{T}} \frac{1}{2\pi} \frac{1}{N+1} \left(\sum_{M=0}^{N} \sum_{m=-M}^{M} e^{im(x-y)} \right) f(y)\, dy \\ &= \int_{\mathbb{T}} K_N(x-y) f(y) dy, \end{aligned}$$

where

$$\begin{aligned} K_N(x) &= \frac{1}{2\pi}\frac{1}{N+1}\left(\sum_{M=0}^{N}\sum_{m=-M}^{M} e^{imx}\right) \\ &= \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=-N}^{N}(N+1-|n|)\,e^{inx}. \end{aligned}$$

- (b) If $x \neq 0$, then, using the sum of a geometric series, we get

$$\begin{aligned} \sum_{n=-N}^{N}(N+1-|n|)\,e^{inx} &= \left[\sum_{n=0}^{N} e^{i\left(n-\frac{N}{2}\right)x}\right]^2 \\ &= \left[e^{-\frac{iNx}{2}}\left(\sum_{n=0}^{N} e^{inx}\right)\right]^2 \\ &= \left[e^{-\frac{iNx}{2}}\left(\frac{1-e^{i(N+1)x}}{1-e^{ix}}\right)\right]^2 \\ &= \left[\frac{\exp\left(-\frac{i(N+1)x}{2}\right)-\exp\left(\frac{i(N+1)x}{2}\right)}{\exp\left(-\frac{ix}{2}\right)-\exp\left(\frac{ix}{2}\right)}\right]^2 \\ &= \left[\frac{\sin\frac{(N+1)x}{2}}{\sin\frac{x}{2}}\right]^2. \end{aligned}$$

The expression for $K_N(x)$ with $x \neq 0$ then follows.

- For $x = 0$, we have

$$K_N(0) = \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=-N}^{N}(N+1-|n|).$$

Using the standard formula for the sum $\sum_{n=1}^{N} n$, we compute that

$$\begin{aligned} \sum_{n=-N}^{N}(N+1-|n|) &= N+1+2\sum_{n=1}^{N}(N+1-n) \\ &= N+1+2\sum_{n=1}^{N} n \end{aligned}$$

$$\begin{aligned} &= N+1+2\cdot\frac{1}{2}N(N+1) \\ &= (N+1)^2. \end{aligned}$$

The expression for $K_N(0)$ then follows.

- (c) It is immediate from the expression in (b) that $K_N \geq 0$.
- Since
$$\int_{\mathbb{T}} e^{inx}\,dx = 0 \quad \text{if } n \neq 0,$$
the only nonzero term in the integration of the expression for K_N in (a) is the one with $n = 0$, so that
$$\int_{\mathbb{T}} K_N(x)\,dx = \frac{1}{2\pi}\frac{1}{N+1}(N+1)\int_{\mathbb{T}} 1\,dx = 1.$$
- If $0 < \delta \leq \pi$ and $\delta \leq |x| \leq \pi$, then from (b) we see that
$$K_N(x) \leq \frac{1}{2\pi}\frac{1}{N+1}\left(\frac{1}{\sin\delta/2}\right)^2.$$
If follows that $K_N \to 0$ uniformly on $\delta \leq |x| \leq \pi$ as $N \to \infty$, so that
$$\lim_{N\to\infty}\int_{\delta\leq|x|\leq\pi} K_N(x)\,dx = 0.$$
This shows that the sequence $(K_N)_{N=1}^{\infty}$ is an approximate identity.
- If $f \in C(\mathbb{T})$ is continuous, then the Fejér sums $f_N = K_N * f$ converge uniformly to f on $\mathbb{T}$.

Remark. One may write the partial sums of the Fourier series of f,

$$S_N(f)(x) = \frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N}\hat{f}_n e^{inx},$$

as a convolution $S_N(f) = D_N * f$ of f with the Dirichlet kernel,

$$\begin{aligned} D_N(x) &= \frac{\sin\left((N+1/2)\,x\right)}{2\pi\sin\left(x/2\right)} \qquad x \neq 0, \\ D_N(0) &= \frac{2N+1}{2\pi}. \end{aligned}$$

The sequence $(D_N)_{N=1}^{\infty}$ is *not* an approximate identity. Although

$$\int_{\mathbb{T}} D_N(x)\,dx = 1 \qquad \text{for every } N \in \mathbb{N},$$

the kernel D_N is not nonnegative; more importantly,

$$\int_{\mathbb{T}} |D_N(x)|\;dx \to \infty \qquad \text{as } N \to \infty.$$

(The integral grows roughly like $\log N$.) As a result, it is *not* true that the partial sums of the Fourier series of a continuous function converge uniformly to the function. Averaging the partial sums, however, leads to the Fejér kernel and a uniformly convergent sequence of trigonometric polynomials. For more discussion of this problem, and many other things, see *Fourier Analysis* by T. W. Körner.