Solutions: Problem Set 6
Math 201B, Winter 2007

Problem 1. Consider the Schrödinger equation on the circle,

$$\begin{aligned} iu_t &= u_{xx}, && x \in \mathbb{T},\ t \in \mathbb{R}, \\ u(x,0) &= f(x), && x \in \mathbb{T}, \end{aligned}$$

where $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $f : \mathbb{T} \to \mathbb{C}$ and the derivatives are interpreted in an appropriate sense.

(a) Solve for $u(x,t)$ by the use of Fourier series. If $U(t) = u(\cdot, t) \in L^2(\mathbb{T})$, show that

$$U(t) = T(t)f$$

where $T(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear operator, defined for all $t \in \mathbb{R}$.

(b) Show that $T(t)$ is a unitary operator.

(c) Briefly compare the qualitative properties (smoothing, reversibility, long-time behavior) of the Schrödinger equation with those of the heat equation.

Solution.

- Writing u in a Fourier series,

  $$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{u}_n(t) e^{inx},$$

  and taking Fourier coefficients of the equation, we get

  $$\begin{aligned} i\frac{d\hat{u}_n}{dt} &= -n^2 \hat{u}_n, \\ \hat{u}_n(0) &= \hat{f}_n. \end{aligned}$$

  The solution is

  $$\hat{u}_n(t) = \hat{f}_n e^{in^2 t}.$$

  It follows that

  $$u(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in^2 t} e^{inx}.$$

- We may write $u(\cdot, t) = T(t)f$ where the operator $T(t)$ is defined by

$$\widehat{(T(t)f)}_n = e^{in^2 t}\hat{f}_n.$$

Note that since $|e^{in^2 t}| = 1$, Parseval's theorem implies that

$$\begin{aligned} \|T(t)f\|^2 &= \sum_{n\in\mathbb{Z}} \left|\widehat{(T(t)f)}_n\right|^2 \\ &= \sum_{n\in\mathbb{Z}} \left|e^{in^2 t}\hat{f}_n\right|^2 \\ &= \sum_{n\in\mathbb{Z}} \left|\hat{f}_n\right|^2 \\ &= \|f\|^2. \end{aligned}$$

Thus, $T(t)f \in L^2(\mathbb{T})$ for every $f \in L^2(\mathbb{T})$, and $T(t)$ is an isometry on $L^2(\mathbb{T})$.

- Similarly if $g \in L^2(\mathbb{T})$, then, since $e^{in^2 t} \neq 0$, there exists a unique $f \in L^2(\mathbb{T})$ such that $Tf = g$, given by

$$\hat{f}_n = e^{-in^2 t}\hat{g}_n.$$

Thus, $T(t)$ is invertible, with

$$T^{-1}(t) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$$

defined by

$$\widehat{(T(t)^{-1}f)}_n = e^{-in^2 t}\hat{f}_n.$$

- It follows from Parseval's theorem that

$$\begin{aligned} \langle f, Tg\rangle &= \sum_{n\in\mathbb{Z}} \overline{\hat{f}_n}\widehat{Tg}_n \\ &= \sum_{n\in\mathbb{Z}} \overline{\hat{f}_n} e^{in^2 t}\widehat{g}_n \\ &= \sum_{n\in\mathbb{Z}} \overline{e^{-in^2 t}\hat{f}_n}\widehat{g}_n \\ &= \langle T^* f, g\rangle, \end{aligned}$$

where

$$\widehat{(T^*(t)f)}_n = e^{-in^2t}\hat{f}_n.$$

We see that $T^*(t) = T^{-1}(t)$, so $T(t)$ is unitary.

- (c) A similar argument to the one above shows that for any $s \geq 0$

$$\sum_{n\in\mathbb{Z}}(1+n^2)^s \left|\widehat{(T(t)f)}_n\right|^2 = \sum_{n\in\mathbb{Z}}(1+n^2)^s \left|\hat{f}_n\right|^2$$

so $T(t)f \in H^s(\mathbb{T})$ if and only if $f \in H^2(\mathbb{T})$. Thus, the solutions at time t has exactly the same smoothness, as measured by the Sobolev spaces $H^s(\mathbb{T})$, as the initial data f, and, unlike the heat equation, the Schrödinger equation does not smooth the solution.

- The Schrödinger equation can be solved both forwards and backwards in time, unlike the heat equation which can be solved only forwards in time.

- Finally, unlike the solution of the heat equation, the solution of the Schrödinger equation does not approach a steady state as $t \to \infty$; instead it is an almost-periodic, oscillatory function of t.

Remark. The Schrödinger equation is a typical example of a dispersive wave equation. This partial differential equation describes a single non-relativistic quantum mechanical particle, which is not subject to any forces, that moves around a one-dimensional circle. The wave-function $u(x,t)$ has the interpretation that $|u(x,t)|^2$ is the spatial probability density of finding the particle at the spatial location x at time t.

Problem 2. (a) Suppose that P, Q are orthogonal projections on a Hilbert space. Prove that $PQ = 0$ if and only if $\operatorname{ran} P \perp \operatorname{ran} Q$.
(b) Suppose that $\{P_1, P_2, \dots, P_n\}$ is a family of orthogonal projections on a Hilbert space, and $P_j P_k = 0$ for $j \neq k$. Prove that $P_1 + P_2 + \dots + P_n$ is an orthogonal projection.
(c) Suppose that $\{P_k \mid k \in \mathbb{N}\}$ is a countably-infinite family of orthogonal projections on a Hilbert space $\mathcal{H}$ such that

$$\bigoplus_{k\in\mathbb{N}} \operatorname{ran} P_k = \mathcal{H}, \qquad P_j P_k = 0 \quad \text{for } j \neq k.$$

Prove that for every $x \in \mathcal{H}$

$$\sum_{k=1}^{\infty} P_k x = x,$$

where the series converges strongly (i.e. with respect to the norm) in $\mathcal{H}$. Is it true or false that

$$\sum_{k=1}^{\infty} P_k = I,$$

where the series converges with respect to the operator norm on $\mathcal{B}(\mathcal{H})$?

Solution.

- (a) If $PQ = 0$, then $\operatorname{ran} Q \subset \ker P$, so $(\ker P)^\perp \subset (\operatorname{ran} Q)^\perp$. Since $(\ker P)^\perp = \operatorname{ran} P$, we see that $\operatorname{ran} P \perp \operatorname{ran} Q$.

- Conversely, if $\operatorname{ran} P \perp \operatorname{ran} Q$, then $\operatorname{ran} P \subset (\operatorname{ran} Q)^\perp$, which implies that $(\operatorname{ran} Q)^{\perp\perp} \subset (\operatorname{ran} P)^\perp$. Since $\operatorname{ran} Q$ is closed, $(\operatorname{ran} Q)^{\perp\perp} = \operatorname{ran} Q$, and since P is an orthogonal projection $(\operatorname{ran} P)^\perp = \ker P$. Hence $\operatorname{ran} Q \subset \ker P$, and $PQ = 0$.

- (b) Let $E = P_1 + \dots + P_n$. Since $P_j^* = P_j$, $P_j^2 = P_j$, and $P_j P_k = 0$ for $j \neq k$, we have

  $$E^* = (P_1 + \dots + P_n)^* = P_1^* + \dots + P_n^* = P_1 + \dots + P_n = E,$$

  and

  $$E^2 = \left(\sum_{j=1}^{n} P_j\right)\left(\sum_{k=1}^{n} P_k\right) = \sum_{j,k=1}^{n} P_j P_k = \sum_{j=1}^{n} P_j^2 = \sum_{j=1}^{n} P_j = E,$$

  so E is an orthogonal projection.

- (c) Let

$$E_n = \sum_{k=1}^{n} P_k.$$

Then E_n is an orthogonal projection, so $\langle x, E_n x\rangle$ is real, and

$$\|E_n x\|^2 = \langle E_n x, E_n x\rangle = \left\langle x, E_n^2 x\right\rangle = \langle x, E_n x\rangle .$$

As in the proof of Bessel's inequality, we compute that

$$\begin{aligned} 0 &\leq \|E_n x - x\|^2 \\ &\leq \langle E_n x - x, E_n x - x\rangle \\ &\leq \|E_n x\|^2 - 2\langle x, E_n x\rangle + \|x\|^2 \\ &\leq \|x\|^2 - \|E_n x\|^2 , \end{aligned}$$

so for every $n \in \mathbb{N}$, we have

$$\|E_n x\|^2 \leq \|x\|^2.$$

- Since the P_k are mutually orthogonal projections, the sequence $(P_k x)$ is orthogonal, and by the Pythagorean theorem

$$\|E_n x\|^2 = \sum_{k=1}^{n} \|P_k x\|^2 .$$

It follows that

$$\sum_{k=1}^{n} \|P_k x\|^2 \leq \|x\|^2,$$

which implies that $\sum_{k=1}^{\infty} P_k x$ converges, to $y \in \mathcal{H}$, say.

- Suppose that $z \in \operatorname{ran} P_k$. Then $P_k z = z$ and $z \in (\operatorname{ran} P_j)^{\perp}$ for $j \neq k$, so

$$\langle z, y\rangle = \left\langle z, \sum_{j=1}^{\infty} P_j x \right\rangle = \langle z, P_k x\rangle = \langle P_k z, x\rangle = \langle z, x\rangle .$$

It follows that $(x - y) \perp \operatorname{ran} P_k$ for every $k \in \mathbb{N}$, which implies that

$$(x - y) \perp \bigoplus_{k \in \mathbb{N}} \operatorname{ran} P_k.$$

Hence $x - y = 0$, and

$$\sum_{k=1}^{\infty} P_k x = x.$$

Problem 3. (a) Suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces. Define $\mathcal{H}_1 \oplus \mathcal{H}_2$ as the linear space of ordered pairs

$$\mathcal{H}_1 \oplus \mathcal{H}_2 = \{(x_1, x_2) \mid x_1 \in \mathcal{H}_1,\ x_2 \in \mathcal{H}_2\},$$

with the inner product of $x, y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, with $x = (x_1, x_2)$, $y = (y_1, y_2)$, defined by

$$\langle x, y \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle x_1, y_1 \rangle_{\mathcal{H}_1} + \langle x_2, y_2 \rangle_{\mathcal{H}_2}.$$

Prove that $\mathcal{H}_1 \oplus \mathcal{H}_2$ is a Hilbert space.

(b) Suppose that $\{\mathcal{H}_\alpha \mid \alpha \in A\}$ is an arbitrary indexed family of Hilbert spaces. Define

$$\bigoplus_{\alpha \in A} \mathcal{H}_\alpha = \left\{ (x_\alpha)_{\alpha \in A} \mid x_\alpha \in \mathcal{H}_\alpha, \quad \sum_{\alpha \in A} \|x_\alpha\|^2 < \infty \right\},$$

with the inner product of

$$x = (x_\alpha) \in \bigoplus_{\alpha \in A} \mathcal{H}_\alpha, \qquad y = (y_\alpha) \in \bigoplus_{\alpha \in A} \mathcal{H}_\alpha$$

defined by

$$\langle x, y \rangle = \sum_{\alpha \in A} \langle x_\alpha, y_\alpha \rangle.$$

Prove that $\bigoplus_{\alpha \in A} \mathcal{H}_\alpha$ is a Hilbert space.

Solution.

- (a) This is straightforward to verify.
- (b) First, we prove that

  $$\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{H}_\alpha$$

  is a linear space. If $\lambda \in \mathbb{C}$ and $x = (x_\alpha) \in \mathcal{H}$, then

  $$\sum_{\alpha \in A} \|\lambda x_\alpha\|^2 = |\lambda|^2 \sum_{\alpha \in A} \|x_\alpha\|^2 < \infty,$$

so $\lambda x \in \mathcal{H}$. If $x = (x_\alpha) \in \mathcal{H}$, $y = (y_\alpha) \in \mathcal{H}$, and $I \subset A$ is a finite subset, then using the triangle inequality in $\mathcal{H}_\alpha$ and the triangle inequality in $\ell^2(I)$,

$$\left(\sum_{\alpha\in I}[a_\alpha + b_\alpha]^2\right)^{1/2} \leq \left(\sum_{\alpha\in I}|a_\alpha|^2\right)^{1/2} + \left(\sum_{\alpha\in I}|b_\alpha|^2\right)^{1/2},$$

we get

$$\begin{aligned}\left(\sum_{\alpha\in I}\|x_\alpha + y_\alpha\|^2\right)^{1/2} &\leq \left(\sum_{\alpha\in A}[\|x_\alpha\| + \|y_\alpha\|]^2\right)^{1/2} \\ &\leq \left(\sum_{\alpha\in I}\|x_\alpha\|^2\right)^{1/2} + \left(\sum_{\alpha\in I}\|y_\alpha\|^2\right)^{1/2}.\end{aligned}$$

It follows that $\sum_{\alpha\in A}\|x_\alpha + y_\alpha\|^2 < \infty$, so $(x + y) \in \mathcal{H}$.

- The series defining the inner product is absolutely convergent and well-defined on $\mathcal{H}$ as an unordered sum since, by the Cauchy-Schwartz inequality, for any finite subset $I \subset A$

$$\begin{aligned}\sum_{\alpha\in I}|\langle x_\alpha, y_\alpha\rangle| &\leq \sum_{\alpha\in I}\|x_\alpha\|\,\|y_\alpha\| \\ &\leq \left(\sum_{\alpha\in I}\|x_\alpha\|^2\right)^{1/2}\left(\sum_{\alpha\in I}\|y_\alpha\|^2\right)^{1/2} \\ &\leq \|x\|\,\|y\|.\end{aligned}$$

  It is straightforward to verify that $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ has the properties of an inner product.

- The main thing we need to prove is that $\mathcal{H}$ is complete. Suppose that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$, with $x_n = (x_{n,\alpha})_{\alpha\in A}$, where $x_{n,\alpha} \in \mathcal{H}_\alpha$. Then, since

$$\|x_{n,\alpha} - x_{m,\alpha}\| \leq \|x_n - x_m\|,$$

  $(x_{n,\alpha})_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}_\alpha$ for each $\alpha \in A$. Since $\mathcal{H}_\alpha$ is complete, there exists $x_\alpha \in \mathcal{H}_\alpha$ such that $x_{n,\alpha} \to x_\alpha$ as $n \to \infty$. Let $x = (x_\alpha)_{\alpha\in A}$. We claim that $\|x - x_n\| \to 0$ as $n \to \infty$ and $x \in \mathcal{H}$, meaning that $\mathcal{H}$ is complete.

- If $I \subset A$ is ay finite subset, then

$$\begin{aligned} \sum_{\alpha \in I} \|x_\alpha - x_{n,\alpha}\|^2 &= \lim_{m\to\infty} \sum_{\alpha \in I} \|x_{m,\alpha} - x_{n,\alpha}\|^2 \\ &\leq \lim_{m\to\infty} \sum_{\alpha \in A} \|x_{m,\alpha} - x_{n,\alpha}\|^2 \\ &= \lim_{m\to\infty} \|x_m - x_n\|^2. \end{aligned}$$

  Since the sequence (x_n) is Cauchy, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \epsilon$ for all $n, m \geq N$. It follows that if $n \geq N$, then $\lim_{m\to\infty} \|x_m - x_n\|^2 \leq \epsilon^2$, and

$$\begin{aligned} \|x - x_n\| &= \left(\sum_{\alpha \in A} \|x_\alpha - x_{n,\alpha}\|^2 \right)^{1/2} \\ &= \sup \left\{ \left(\sum_{\alpha \in I} \|x_\alpha - x_{n,\alpha}\|^2 \right)^{1/2} \mid I \subset A \text{ finite} \right\} \\ &\leq \epsilon, \end{aligned}$$

  meaning that $\|x - x_n\| \to 0$ as $n \to \infty$.

- By the previous proof, we can pick $n \in \mathbb{N}$ such that $\|x - x_n\| \leq 1$, meaning that $x - x_n \in \mathcal{H}$. Then $x = (x - x_n) + x_n \in \mathcal{H}$ since $\mathcal{H}$ is closed under addition.

Remark. A special case of this proof is the fact that

$$\ell^2(\mathbb{N}) = \bigoplus_{n \in \mathbb{N}} \mathbb{C}$$

is a Hilbert space.