Solutions: Problem Set 7
Math 201B, Winter 2007

Problem 1. Suppose that $m : [0,1] \to \mathbb{C}$ is a continuous complex-valued function on $[0,1]$. Define a multiplication operator

$$M : L^2([0,1]) \to L^2([0,1])$$

by

$$(Mf)(x) = m(x)f(x).$$

(a) Prove that M is a bounded linear operator on $L^2([0,1])$ and compute its adjoint M^*.
(b) For what functions m is M: (i) self-adjoint; (ii) skew-adjoint; (iii) unitary?

Solution.

- (a) We have

$$\begin{aligned}
\|Mf\| &= \left(\int_0^1 |Mf(x)|^2\, dx\right)^{1/2} \\
&= \left(\int_0^1 |m(x)f(x)|^2\, dx\right)^{1/2} \\
&\leq \left(\sup_{x\in[0,1]} |m(x)|\right)\left(\int_0^1 |f(x)|^2\, dx\right)^{1/2} \\
&\leq \|m\|_\infty \|f\|,
\end{aligned}$$

  where $\|m\|_\infty = \sup_{x\in[0,1]} |m(x)|$ is finite since a continuous function on a compact set is bounded. It follows that $M : L^2([0,1]) \to L^2([0,1])$ is bounded and

$$\|M\| \leq \|m\|_\infty.$$

  In fact, $\|M\| = \|m\|_\infty$, as can be seen by considering the action of M on functions that are supported in a small interval about a point where $|m|$ attains its maximum.

- For $f, g \in L^2([0,1])$, we have

$$\begin{aligned}
\langle M^* f, g \rangle &= \langle f, Mg \rangle \\
&= \int_0^1 \overline{f(x)} Mg(x)\, dx \\
&= \int_0^1 \overline{f(x)} m(x) g(x)\, dx \\
&= \int_0^1 \overline{\overline{m(x)} f(x)} g(x)\, dx.
\end{aligned}$$

  Thus

$$M^* f(x) = \overline{m(x)} f(x),$$

  and M^* is multiplication by the complex-conjugate of m.

  (b) The multiplication operator M is self-adjoint if m is real-valued, skew-adjoint if m is imaginary-valued, and unitary if m takes values in the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Note that M and M^* commute, so any multiplication operator is normal.

Problem 2. The Hilbert transform $H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by

$$H\left(\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{inx}\right) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}} i\,\operatorname{sgn} n\,\hat{f}(n)e^{inx},$$

where

$$\operatorname{sgn} n = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n < 0. \end{cases}$$

That is, the Hilbert transform acts on a function by multiplying its nth Fourier coefficient by i if n is positive and $-i$ if n is negative.

(a) If $n \in \mathbb{N}$ is a positive integer, compute $H(\cos nx)$ and $H(\sin nx)$. Show that H is a bounded linear map on $L^2(\mathbb{T})$ and compute its norm.

(b) Show that H is skew-adjoint.

(c) Let $\mathcal{M}$ be the subspace of periodic functions with zero mean,

$$\mathcal{M} = \left\{ f \in L^2(\mathbb{T}) \mid \int_{\mathbb{T}} f\,dx = 0 \right\}.$$

Show that the range of H is equal to $\mathcal{M}$. What is the kernel of H?

(d) Show that $H^2 = -I$ on $\mathcal{M}$ and that H is a unitary transformation on $\mathcal{M}$.

Solution.

- (a) For $n \in \mathbb{Z}$, we have

$$H\left(e^{inx}\right) = \begin{cases} ie^{inx} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -ie^{inx} & \text{if } n < 0. \end{cases}$$

  It follows that if $n \in \mathbb{N}$, then

$$\begin{aligned} H(\cos nx) &= H\left(\frac{e^{inx} + e^{-inx}}{2}\right) \\ &= \frac{ie^{inx} - ie^{-inx}}{2} \\ &= -\sin nx, \end{aligned}$$

$$
\begin{aligned}
H(\sin nx) &= H\left(\frac{e^{inx}-e^{-inx}}{2i}\right) \\
&= \frac{e^{inx}+e^{-inx}}{2} \\
&= \cos nx.
\end{aligned}
$$

- By Parseval's theorem,

$$
\begin{aligned}
\|Hf\|^2 &= \sum_{n\in\mathbb{Z}} \left|\widehat{Hf}(n)\right|^2 \\
&= \sum_{n\in\mathbb{Z}} \left| i\operatorname{sgn} n \widehat{f}(n)\right|^2 \\
&= \sum_{n\neq 0} \left|\widehat{f}(n)\right|^2 \\
&\leq \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|^2 \\
&\leq \|f\|^2,
\end{aligned}
$$

  so H is bounded with $\|H\| \leq 1$. Since $\|He^{inx}\| = \|e^{inx}\|$, we see that $\|H\| = 1$.

- (b) By Parseval's theorem, if $f, g \in L^2(\mathbb{T})$, then

$$
\begin{aligned}
\langle f, Hg\rangle &= \int_{\mathbb{T}} \overline{f(x)}\, Hg(x)\, dx \\
&= \sum_{n\in\mathbb{Z}} \overline{\widehat{f}(n)}\, \widehat{Hg}(n) \\
&= \sum_{n\in\mathbb{Z}} \overline{\widehat{f}(n)}\, i\operatorname{sgn} n \widehat{g}(n) \\
&= -\sum_{n\in\mathbb{Z}} \overline{i\operatorname{sgn} n \widehat{f}(n)}\, \widehat{g}(n) \\
&= -\langle Hf, g\rangle,
\end{aligned}
$$

  so $H^* = -H$.

- (c) If $g = Hf$, then $\hat{g}(0) = 0$. Since

$$
\hat{g}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} g(x)\, dx,
$$

it follows that $g \in \mathcal{M}$, so $\operatorname{ran} H \subset \mathcal{M}$.

- Conversely, if $g \in \mathcal{M}$ then

$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} \hat{g}(n) e^{inx},$$

where

$$\|g\|^2 = \sum_{n \neq 0} |\hat{g}(n)|^2 < \infty.$$

Since $1/(i\operatorname{sgn} n) = -i\operatorname{sgn} n$ for $n \neq 0$, it follows that $g = Hf \in \operatorname{ran} H$ where $f = -Hg \in \mathcal{M}$ is given by

$$f(x) = -\frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} i\operatorname{sgn} n \hat{g}(n) e^{inx}.$$

Thus, $\operatorname{ran} H = \mathcal{M}$, and $H^2 = -I$ on $\mathcal{M}$.

- The kernel of H is

$$\begin{aligned} \ker H &= \left\{ f \in L^2(\mathbb{T}) \mid \hat{f}(n) = 0 \text{ for } n \neq 0 \right\} \\ &= \{\text{constant functions on } \mathbb{T}\}. \end{aligned}$$

- (d) We have shown that $H^2 = -I$ on $\mathcal{M}$ and $H^* = -H$ on $L^2(\mathbb{T})$. It follows that $H^{-1} = -H = H^*$ on $\mathcal{M}$, so H is unitary on $\mathcal{M}$.

Problem 3. Let $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$ be the Hilbert spaces of periodic square-integrable functions and functions with square-integrable weak derivatives, respectively, with the inner products

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{T}} \overline{f} g \, dx, \qquad \langle f, g\rangle_{H^1} = \int_{\mathbb{T}} \left(\overline{f} g + \overline{f'} g'\right) \, dx.$$

Let $D : H^1(\mathbb{T}) \to L^2(\mathbb{T})$ be the derivative operator $D = d/dx$ defined by

$$\widehat{(Df)}(n) = in\widehat{f}(n).$$

Prove that $D^* : L^2(\mathbb{T}) \to H^1(\mathbb{T})$ is given by

$$D^* = D\left(D^2 - 1\right)^{-1}.$$

Solution.

- From Parseval's theorem, we have

$$\begin{aligned}
\langle f, g\rangle_{L^2} &= \sum_{n\in\mathbb{Z}} \overline{\hat{f}(n)}\hat{g}(n) \\
\langle f, g\rangle_{H^1} &= \sum_{n\in\mathbb{Z}} \left(1+n^2\right) \overline{\hat{f}(n)}\hat{g}(n)
\end{aligned}$$

- The definition of the adjoint implies that for every $f \in L^2(\mathbb{T})$ and $g \in H^1(\mathbb{T})$, we have

$$\begin{aligned}
\langle f, Dg\rangle_{L^2} &= \sum_{n\in\mathbb{Z}} \overline{\hat{f}(n)}\widehat{Dg}(n) \\
&= \sum_{n\in\mathbb{Z}} \overline{\hat{f}(n)}\, in\hat{g}(n) \\
&= -\sum_{n\in\mathbb{Z}} \overline{in\hat{f}(n)}\hat{g}(n) \\
&= -\sum_{n\in\mathbb{Z}} \left(1+n^2\right) \overline{\left(\frac{in}{n^2+1}\hat{f}(n)\right)}\hat{g}(n) \\
&= \langle D^* f, g\rangle_{H^1},
\end{aligned}$$

where
$$\widehat{(D^* f)}(n) = -\frac{in}{n^2+1}\hat{f}(n).$$
Since the application of D to f corresponds to the multiplication of the nth Fourier coefficient $\hat{f}(n)$ of f by in, we see that
$$D^* = -D(-D^2+1)^{-1},$$
which proves the result.

- Note that $(-D^2+1) : H^2(\mathbb{T}) \to L^2(\mathbb{T})$, given by
$$(\widehat{-D^2+1)f}(n) = \left(n^2+1\right)\hat{f}(n),$$
is invertible, with inverse $(-D^2+1)^{-1} : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ given by
$$(\widehat{-D^2+1)^{-1}f}(n) = \frac{1}{n^2+1}\hat{f}(n),$$
and $D : H^2(\mathbb{T}) \to H^1(\mathbb{T})$ is given by
$$\widehat{(Df)}(n) = in\widehat{f}(n).$$
Thus, the expression for D^* makes sense as a composition of maps. Alternatively, and more simply, one can define the action of D^* on f as multiplication of the nth Fourier coefficient $\hat{f}(n)$ of f by $-in/(n^2+1)$.