Solutions: Problem Set 8
Math 201B, Winter 2007

Problem 1. Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map whose matrix with respect to the standard basis on $\mathbb{C}^2$ is

$$[H] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Why must e^{-itH} be unitary for any $t \in \mathbb{R}$? Compute the matrix of e^{-itH} and verify explicitly that it is unitary.

Solution.

- Since $(e^A)^* = e^{A^*}$, $(e^A)^{-1} = e^{-A}$, and $H = H^*$, we have
$$\left(e^{-itH}\right)^* = e^{(-itH)^*} = e^{itH} = \left(e^{-itH}\right)^{-1},$$
so e^{-itH} is unitary.

- Since $H^2 = I$, we have $H^n = H$ is n is odd and $H^n = I$ if n is even. It follows that
$$\begin{aligned} e^{-itH} &= I + (-itH) + \frac{1}{2!}(-itH)^2 + \frac{1}{3!}(-itH)^3 + \frac{1}{4!}(-itH)^4 + \dots \\ &= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots\right) I - i\left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right) H \\ &= (\cos t)I - i(\sin t)H. \end{aligned}$$
Hence,
$$\left[e^{-itH}\right] = \begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}.$$

- Explicitly, we have
$$\begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix}^* = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix},$$
and
$$\begin{pmatrix} \cos t & -i\sin t \\ -i\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
so $\left[e^{-itH}\right]$ is a unitary matrix.

Problem 2. Define the right and left shift operators S and T on $\ell^2(\mathbb{N})$ by

$$S(x_1, x_2, x_3, x_4, \ldots) = (0, x_1, x_2, x_3, \ldots),$$
$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, x_5, \ldots).$$

(a) Show that $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all $x, y \in \ell^2(\mathbb{N})$. Is S a unitary map on $\ell^2(\mathbb{N})$?
(b) Show that $S^* = T$.
(c) Determine the range and kernel of S and T. Show that both operators have closed range and verify explicitly that

$$\ell^2(\mathbb{N}) = \operatorname{ran} S \oplus \ker S^* = \operatorname{ran} T \oplus \ker T^*.$$

(d) Given any $y \in \ell^2(\mathbb{N})$, find all solutions $x \in \ell^2(\mathbb{N})$, if any, of the equations: (i) $Sx = y$; (ii) $Tx = y$. Do S, T satisfy the Fredholm alternative?

Solution.

- (a) If $x = (x_n)$, $y = y_n$, then

  $$\langle Sx, Sy\rangle = 0 + \overline{x_1}y_1 + \overline{x_2}y_2 + \overline{x_3}y_3 + \ldots = \langle x, y\rangle,$$

  so S preserves inner-products and norms. It is not unitary, however, because it is not onto. Equivalently, we have $S^*S = I$, but not $SS^* = I$. (Note that S, T are not normal operators.)

- (b) We compute that

  $$\langle x, Sy\rangle = \sum_{n=2}^{\infty} \overline{x_n} y_{n-1} = \sum_{n=1}^{\infty} \overline{x_{n+1}} y_n = \langle Tx, y\rangle.$$

  Hence, $S^* = T$. It also follows that $T^* = S$.

- (c) We have

  $$\operatorname{ran} S = \left\{(y_1, y_2, y_3, \ldots) \in \ell^2(\mathbb{N}) \mid y_1 = 0\right\}, \quad \ker S = \{0\},$$
  $$\operatorname{ran} T = \ell^2(\mathbb{N}), \qquad \ker T = \{\lambda e_1 \mid \lambda \in \mathbb{C}\},$$

  where $e_1 = (1, 0, 0, 0, \ldots)$. It is clear that $\operatorname{ran} S = (\ker T)^\perp$, so $\operatorname{ran} S$ is closed and $\ell^2(\mathbb{N}) = \operatorname{ran} S \oplus \ker S^*$. Also, since $\operatorname{ran} T = \ell^2(\mathbb{N})$ and $(\ker T)^* = \{0\}$, the direct sum $\ell^2(\mathbb{N}) = \operatorname{ran} T \oplus \ker T^*$ is trivial.

- The operator S does not satisfy the Fredholm alternative because a solution of $S^*x = 0$ is not unique, but a solution of $Sx = y$ is unique, if it exists at all. The operator T does not satisfy the Fredholm alternative because the only solution of $T^*x = 0$ is $x = 0$, but a solution of $Tx = y$ is not unique.

Remark. For any map $A : \mathbb{C}^n \to \mathbb{C}^n$, the rank (dimension of the range) of A^* is the same as the rank of A ("row-rank = column rank"), and the nullity (dimension of the nullspace) of A is the same as the nullity of A^*. In particular, A one-to-one if and only if it is onto. As illustrated by S, T these results do not hold for general bounded linear maps on infinite-dimensional spaces, even ones with closed range and finite-dimensional nullspaces.

Problem 3. For $n \in \mathbb{N}$, define the following functions $f_n, g_n, h_n : \mathbb{R} \to \mathbb{R}$:

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise;} \end{cases}$$

$$h_n(x) = \begin{cases} 1 & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that none of the sequences (f_n), (g_n), (h_n) converge strongly in $L^2(\mathbb{R})$. Which sequences converge weakly?

Solution.

- The sequence (f_n) converges weakly to 0. If $\phi : \mathbb{R} \to \mathbb{C}$ is a continuous function with compact support, then

$$\begin{aligned} \langle f_n, \phi \rangle &= \sqrt{n} \int_0^{1/n} \phi(x)\, dx \\ &= \frac{1}{\sqrt{n}}\, n \int_0^{1/n} \phi(x)\, dx \\ &\to 0 \qquad \text{as } n \to \infty, \end{aligned}$$

  since, by continuity,

$$n \int_0^{1/n} \phi_n(x)\, dx \to \phi(0) \qquad \text{as } n \to \infty.$$

  We also have that $\|f_n\| = 1$ for every $n \in \mathbb{N}$, so the sequence is bounded. Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we conclude that $f_n \rightharpoonup 0$ as $n \to \infty$.

- If the sequence (f_n) converged strongly, it would have to converge to the weak limit 0. Since $\|f_n\| = 1$ for all $n \in \mathbb{N}$, it does not converge strongly to 0, so the sequence does not converge strongly. (Alternatively, it is easy to check that the sequence is not Cauchy.)

- Computing the L^2-norm of g_n, we get

$$\|g_n\| = \sqrt{n}.$$

  Since the sequence is unbounded, it cannot converge weakly (or strongly).

- If $\phi \in C_c(\mathbb{R})$ then $[n, n+1]$ is disjoint from the support of ϕ for all sufficiently large n, and therefore

$$\langle h_n, \phi \rangle = \int_n^{n+1} \phi(x)\, dx \to 0 \qquad \text{as } n \to \infty.$$

  Also, $\|h_n\| = 1$ for every $n \in \mathbb{N}$, so the sequence is bounded. It follows that $h_n \rightharpoonup 0$. The functions $\{h_n\}$ form an orthonormal set, with $\|h_n - h_m\| = \sqrt{2}$ for $n \neq m$, so the sequnce is not Cauchy and does not converge strongly.

Remark. This problem illustrate two other typical ways, in addition to 'oscillation' discussed in class, for a sequence of functions to converge weakly but not strongly — 'concentration', for (f_n), and 'escape to infinity' for (h_n).