Notes on Unordered Sums

John K. Hunter

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Unconditionally and absolutely convergent series

To motivate the definition of unordered sums, we first consider the convergence of series. If $(x_n)_{n=1}^{\infty}$ is a sequence in a normed space X, then the series

$$\sum_{n=1}^{\infty} x_n \tag{1}$$

converges to $x \in X$ if

$$\left\| \sum_{n=1}^{N} x_n - x \right\| \to 0 \qquad \text{as } N \to \infty.$$

In general, the convergence and sum of a series depend on the order of its terms.

A series $\sum y_m$ is a *rearrangement* of $\sum x_n$ if $y_m = x_{\sigma(m)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is a one-to-one, onto mapping. For example,

$$x_2 + x_1 + x_6 + x_4 + x_3 + x_{12} + x_{10} + x_8 + x_5 + \dots$$

is a rearrangement of $x_1 + x_2 + x_3 + \dots$, but

$$x_1 + x_3 + x_5 + \dots + x_2 + x_4 + x_6 + \dots$$

is not.

A series *converges unconditionally* if every rearrangement of the series converges to the same sum. If a series converges but does not converge unconditionally, then we say that it *converges conditionally*.

1 Example The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges in $\mathbb{R}$ (by the alternating series test), but does not converge unconditionally. For example, we may rearrange the series as

$$\begin{aligned}
&1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}+\frac{1}{7}-\dots\\
&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{5}-\frac{1}{2}\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\frac{1}{7}+\dots,
\end{aligned}$$

which diverges to $-\infty$.

The series (1) *converges absolutely* if

$$\sum_{n=1}^{\infty}\|x_n\|$$

converges in $\mathbb{R}$. An absolutely convergent series in a Banach space is unconditionally convergent (as we show below).

For series in $\mathbb{R}$, or $\mathbb{R}^n$, Riemann proved the converse result that an unconditionally convergent series is absolutely convergent. In fact, if a convergent real series is not absolutely convergent, then given any

$$-\infty\le m\le M\le\infty$$

there is a rearrangement (y_n) of the series such that

$$\liminf_{n\to\infty} y_n=m,\quad \limsup_{n\to\infty} y_n=M.$$

Thus, absolute convergence and unconditional convergence are equivalent in finite-dimensional Banach spaces. In infinite-dimensional Banach spaces, however, a series may converge unconditionally but not absolutely. (Dvoretzky and Rogers proved that *every* infinite-dimensional Banach space contains an unconditionally convergent series that does not converge absolutely.)

2 Example Consider the Hilbert space

$$\ell^2(\mathbb{N})=\left\{(c_n)_{n=1}^{\infty}\mid c_n\in\mathbb{C},\quad \sum_{n=1}^{\infty}|c_n|^2<\infty\right\}$$

of square-summable complex sequences with the usual inner product,

$$\langle (a_n),(b_n)\rangle=\sum_{n=1}^{\infty}\overline{a}_n b_n.$$

Let $x_n = e_n/n \in \ell^2(\mathbb{N})$, where e_n is the nth standard basis element, so that

$$\begin{aligned} x_1 &= (1, 0, 0, 0, \ldots), \\ x_2 &= (0, 1/2, 0, 0, \ldots), \\ x_3 &= (0, 0, 1/3, 0, \ldots). \end{aligned}$$

Then $\sum x_n$ converges unconditionally to

$$x = (1, 1/2, 1/3, 1/4, \ldots) \in \ell^2(\mathbb{N}).$$

To prove this, suppose $\sum y_m$ is a rearrangement of $\sum x_n$ with $y_m = x_{\sigma(m)}$. Since

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \epsilon^2.$$

Let

$$M = \max\{\sigma(1), \sigma(2), \ldots, \sigma(N)\}.$$

The vectors x_n are pairwise orthogonal, with $\|x_n\| = 1/n$, and the terms $y_1, \ldots, y_M$ of the rearranged series include the terms $x_1, \ldots, x_N$ of the original series. It follows that if $M' \geq M$, then

$$\begin{aligned} \left\| x - \sum_{m=1}^{M'} y_m \right\| &= \left\| \sum_{\sigma(n) > M'} x_n \right\| \\ &= \left(\sum_{\sigma(n) > M'} \|x_n\|^2 \right)^{1/2} \\ &\leq \left(\sum_{n > N}^{\infty} \frac{1}{n^2} \right)^{1/2} \\ &< \epsilon. \end{aligned}$$

Hence, $\sum y_m = x$.

This argument also shows that if J is any finite subset of $\mathbb{N}$ such that $\{1, 2, \ldots, N\} \subset J$, then

$$\left\| \sum_{n \in J} x_n - x \right\| < \epsilon,$$

which means that

$$\sum_{n \in \mathbb{N}} x_n = x$$

in the sense of unordered sums defined below.

On the other hand, the series $\sum x_n$ does not converge absolutely, since $\|x_n\| = 1/n$ and

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Conditional and unconditional bases

The distinction between conditional and unconditional convergence is significant in connection with topological bases. (This discussion is not needed below, so you can omit it without loss of continuity.)

A sequence (e_n) is a Schauder basis of a Banach space X if every $x \in X$ has a unique expansion of the form

$$x = \sum_{n=1}^{\infty} c_n e_n \tag{2}$$

for suitable scalars c_n. In general, this series converges conditionally. Therefore the order of the elements in a Schauder basis is important, and a Schauder basis is a conditional basis. If the series in (2) converges unconditionally for every $x \in X$, then we say that $\{e_n\}$ is an unconditional basis.

For example, the standard basis

$$e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, \ldots), \quad e_3 = (0, 0, 1, \ldots), \quad \ldots$$

is an unconditional basis of the Banach space c_0 of sequences (x_n) such that $x_n \to 0$ as $n \to \infty$ equipped with the sup-norm. The Banach space $C([0, 1])$ has a Schauder basis but one can prove that it has no unconditional basis.

A much harder result, proved by Enflo, is that there exist separable Banach spaces with no Schauder basis.

Hilbert spaces are much simpler than Banach space in this respect. A separable Hilbert space has an unconditional basis, since any orthonormal basis is unconditional (as can be seen by essentially the same argument as the one in the example above). An infinite-dimensional, separable Hilbert space also possesses non-orthogonal conditional bases and unconditional bases.

3 Example The set $\{e^{inx} \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2([-\pi, \pi])$, after suitable normalization. Kadets proved that if $(\lambda_n)_{n\in\mathbb{Z}}$ is any sequence of real numbers such that

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| < \frac{1}{4},$$

meaning that (λ_n) is 'close enough' to the orthogonal case (n), then the set $\left\{e^{i\lambda_n x} \mid n \in \mathbb{Z}\right\}$ is an unconditional basis of $L^2([-\pi, \pi])$.

Unordered sums

Suppose that A is a nonempty set and $\{x_\alpha \mid \alpha \in A\}$ is an indexed set in a normed space X, indexed by A. An *indexed set*

$$\{x_\alpha \in X \mid \alpha \in A\}$$

is a convenient way to write a function $f : A \to X$, where $x_\alpha = f(\alpha)$. For example, a sequence $(x_n)_{n=1}^{\infty}$ is an indexed set $\{x_n \mid n \in \mathbb{N}\}$.

Note that, in general, we may have $x_\alpha = x_\beta$ for $\alpha \neq \beta$. Thus, despite the somewhat ambiguous notation, an indexed set is not the same thing as a set, which is determined entirely by its elements.

Our aim is to define an *unordered sum*

$$\sum_{\alpha\in A} x_\alpha$$

as a limit of finite partial sums, independently of the way in which the terms x_α in the sum are ordered. We allow the index set A to have any cardinality, although we shall see that in a normed space only countably many terms of a convergent unordered sum can be non-zero.

A binary relation $\prec$ on a set $\mathcal{F}$ is a *partial order* on $\mathcal{F}$ if for all $I, J, K \in \mathcal{F}$:

- $I \prec I$;
- if $I \prec J$ and $J \prec I$, then $I = J$;
- if $I \prec J$ and $J \prec K$, then $I \prec K$.

We denote by $\mathcal{F}$ the collection of all finite subsets of the index set A. For $I, J \in \mathcal{F}$, we write $I \prec J$, or $J \succ I$, if $I \subset J$. The symbols '$\prec$' and '$\subset$' are synonymous here; we use '$\prec$' simply to indicate that we are considering '$\subset$' as a partial order on the finite subsets of A.

When A has at least two elements, the relation $\prec$ is not a total order on $\mathcal{F}$ since there are finite subsets I, J of A such that $I \not\subset J$ and $J \not\subset I$.

If $I, J \in \mathcal{F}$, then there exists $K \in \mathcal{F}$ such that $I \prec K$ and $J \prec K$. For example, we can use $K = I \cup J$. A partially ordered set with this property is called a *directed set*.

Given an indexed set $\{x_\alpha \mid \alpha \in A\}$ of terms in X, indexed by A, we define the indexed set $\{S_I \mid I \in \mathcal{F}\}$ of finite partial sums in X, indexed by the collection $\mathcal{F}$ of finite subsets of A, where

$$S_I = \sum_{\alpha \in I} x_\alpha.$$

The mapping $I \mapsto S_I$ from $\mathcal{F}$ into X is an example of a net. (A *net* is a generalization of a sequence that is indexed by a directed set instead of by the natural numbers.) The following definition of the convergence of an unordered sum is a special case of the definition of the convergence of nets.

4 Definition An unordered sum $\sum_{\alpha \in A} x_\alpha$ in a normed space X *converges* to $x \in X$ if for every $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \prec J$ we have

$$\left\| x - \sum_{\alpha \in J} x_\alpha \right\| < \epsilon.$$

5 Example Consider $A = \mathbb{N}$. The unordered sum $\sum_{n \in \mathbb{N}} x_n$ in a normed linear space converges if and only if the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally. The proof is left as an exercise.

There is a natural definition of a Cauchy condition for unordered sums.

6 Definition An unordered sum $\sum_{\alpha \in A} x_\alpha$ in a normed space X is *Cauchy* if for every $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that for every $J, K \in \mathcal{F}$ with $I \prec J$,

$I \prec K$ we have

$$\left\| \sum_{\alpha \in J} x_\alpha - \sum_{\alpha \in K} x_\alpha \right\| < \epsilon. \tag{3}$$

This definition says, roughly, that all 'big enough' partial sums are arbitrarily close. The following theorem gives a useful equivalent form of the Cauchy condition in a normed space, which says, roughly, that the sums of all 'small enough' tails are arbitrarily small.

7 Theorem An unordered sum $\sum_{\alpha \in A} x_\alpha$ in a normed space X is Cauchy if and only if for every $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that for every $J \in \mathcal{F}$ with $I \cap J = \emptyset$, we have

$$\left\| \sum_{\alpha \in J} x_\alpha \right\| < \epsilon. \tag{4}$$

Proof. Suppose the unordered sum is Cauchy. Given $\epsilon > 0$, pick $I \in \mathcal{F}$ such that (3) holds whenever $I \subset J$, $I \subset K$. If $I \cap J = \emptyset$, then

$$\left\| \sum_{\alpha \in J} x_\alpha \right\| = \left\| \sum_{\alpha \in I \cup J} x_\alpha - \sum_{\alpha \in I} x_\alpha \right\| < \epsilon,$$

so the condition in (4) holds.

Conversely, suppose that the condition in (4) holds. Given $\epsilon > 0$ pick $I \in \mathcal{F}$ such that if $J \in \mathcal{F}$ and $I \cap J = \emptyset$ then

$$\left\| \sum_{\alpha \in J} x_\alpha \right\| < \frac{\epsilon}{2}.$$

If $J, K \in \mathcal{F}$ and $I \subset J$, $I \subset K$, then

$$\begin{aligned} \left\| \sum_{\alpha \in J} x_\alpha - \sum_{\alpha \in K} x_\alpha \right\| &= \left\| \sum_{\alpha \in J \setminus I} x_\alpha - \sum_{\alpha \in K \setminus I} x_\alpha \right\| \\ &\leq \left\| \sum_{\alpha \in J \setminus I} x_\alpha \right\| + \left\| \sum_{\alpha \in K \setminus I} x_\alpha \right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which proves that the unordered sum is Cauchy. $\square$

One consequence of this result is that a Cauchy unordered sum in a normed space has only countably many non-zero terms.

8 Proposition If $\sum_{\alpha\in A} x_\alpha$ is a Cauchy unordered sum in a normed space, then $\{\alpha \in A \mid x_\alpha \neq 0\}$ is countable.

Proof. For each $n \in \mathbb{N}$, there exists $I_n \in \mathcal{F}$ such that if $J \in \mathcal{F}$ and $I_n \cap J = \emptyset$, then

$$\left\| \sum_{\alpha\in J} x_\alpha \right\| < \frac{1}{n}.$$

Since I_n is finite, the set

$$B = \bigcup_{n=1}^{\infty} I_n$$

is countable. If $\alpha \in A \setminus B$, then $I_n \cap \{\alpha\} = \emptyset$, so $\|x_\alpha\| < 1/n$ for every $n \in \mathbb{N}$. Hence $\|x_\alpha\| = 0$ and $x_\alpha = 0$. $\square$

As in the case of series, it is immediate that convergent sums are Cauchy.

9 Proposition A convergent unordered sum in a normed space is Cauchy.

Proof. Suppose that the unordered sum $\sum_{\alpha\in A} x_\alpha$ converges to x. Given $\epsilon > 0$, pick $I \in \mathcal{F}$ such that if $I \prec J$ then

$$\left\| x - \sum_{\alpha\in J} x_\alpha \right\| < \frac{\epsilon}{2}.$$

If $I \prec J$ and $I \prec K$, then

$$\left\| \sum_{\alpha\in J} x_\alpha - \sum_{\alpha\in K} x_\alpha \right\| \leq \left\| \sum_{\alpha\in J} x_\alpha - x \right\| + \left\| x - \sum_{\alpha\in K} x_\alpha \right\| < \epsilon,$$

so the unordered sum is Cauchy. $\square$

In a complete space, the convergent and Cauchy unordered sums coincide.

10 Theorem An unordered sum in a Banach space converges if and only if it is Cauchy.

Proof. We only need to prove that a Cauchy sum converges. Suppose that the unordered sum $\sum_{\alpha\in A} x_\alpha$ is Cauchy. For $n \in \mathbb{N}$, we choose $\tilde{I}_n$ such that if $\tilde{I}_n \prec J$, $\tilde{I}_n \prec K$ then

$$\left\| \sum_{\alpha\in J} x_\alpha - \sum_{\alpha\in K} x_\alpha \right\| < \frac{1}{n}. \tag{5}$$

We then choose $I_n \in \mathcal{F}$ such that $\tilde{I}_n \prec I_n$ and $I_m \prec I_n$ for every $m \leq n \in \mathbb{N}$; for example,

$$I_n = \bigcup_{k=1}^{n} \tilde{I}_k.$$

This gives an increasing sequence $I_1 \prec I_2 \prec I_3 \prec \dots$ of finite subsets of A such that (5) holds if $I_n \prec J$, $I_n \prec K$.

For $n \in \mathbb{N}$, we let

$$s_n = \sum_{\alpha\in I_n} x_\alpha.$$

Then $\|s_j - s_k\| < 1/n$ for $j, k \geq n$, so (s_n) is a Cauchy sequence in X. Since X is complete, it converges, to x say.

To show that the unordered sum converges, let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\|x - s_n\| < \epsilon/2$ and $1/n < \epsilon/2$. If $I_n \prec J$, then by the convergence of the sequence (s_n) and the unordered Cauchy property (5), we have

$$\begin{aligned}
\left\| x - \sum_{\alpha\in J} x_\alpha \right\| &\leq \left\| x - \sum_{\alpha\in I_n} x_\alpha \right\| + \left\| \sum_{\alpha\in I_n} x_\alpha - \sum_{\alpha\in J} x_\alpha \right\| \\
&< \|x - s_n\| + \frac{1}{n} \\
&< \epsilon,
\end{aligned}$$

which proves that the unordered sum converges to x. $\square$

We have the following continuity of the inner-product in a Hilbert space with respect to unordered sums, whose proof we leave as an exercise.

11 Theorem Suppose that the unordered sums

$$\sum_{\alpha\in A} x_\alpha, \qquad \sum_{\beta\in B} y_\beta$$

converge in a Hilbert space. Then the unordered sum

$$\sum_{(\alpha,\beta)\in A\times B} \langle x_\alpha, y_\beta \rangle$$

converges in $\mathbb{C}$, and

$$\left\langle \sum_{\alpha\in A} x_\alpha, \sum_{\beta\in B} y_\beta \right\rangle = \sum_{(\alpha,\beta)\in A\times B} \langle x_\alpha, y_\beta \rangle .$$

Unordered sums of non-negative real numbers

A series of non-negative numbers whose partial sums are bounded from above converges to the supremum of its partial sums. The same result holds for unordered sums.

12 Proposition Suppose that $\{x_\alpha \mid \alpha \in A\}$ is an indexed set of non-negative real numbers $x_\alpha \geq 0$. Define $0 \leq M \leq \infty$ by

$$M = \sup \left\{ \sum_{\alpha\in I} x_\alpha \mid I \in \mathcal{F} \right\}.$$

The unordered sum $\sum_{\alpha\in A} x_\alpha$ converges if and only if M is finite, and then the sum is equal to M.

Proof. Since $x_\alpha \geq 0$, if $I \prec J$ then

$$\sum_{\alpha\in I} x_\alpha \leq \sum_{\alpha\in J} x_\alpha \leq M.$$

If $M = \infty$, it follows that $\sum_{\alpha\in I} x_\alpha$ is arbitrarily large for all sufficiently large sets $I \in \mathcal{F}$, so the unordered sum cannot converge.

If $0 \leq M < \infty$, then, by the definition of the supremum, given any $\epsilon > 0$ there exists $I \in \mathcal{F}$ such that

$$M - \epsilon < \sum_{\alpha\in I} x_\alpha \leq M.$$

Thus, if $I \prec J$, then

$$\left| \sum_{\alpha \in J} x_\alpha - M \right| < \epsilon,$$

which proves that the unordered sum converges to M. $\square$

We say that an unordered sum $\sum_{\alpha \in A} x_\alpha$ in a Banach space X *converges absolutely* if the unordered sum $\sum_{\alpha \in A} \|x_\alpha\|$ converges in $\mathbb{R}$. Since

$$\left\| \sum_{\alpha \in I} x_\alpha \right\| \le \sum_{\alpha \in I} \|x_\alpha\|$$

Theorem 7 implies that an absolutely convergent unordered sum is Cauchy, and hence convergent by Theorem 10.

In general $\sum_{\alpha \in I} \|x_\alpha\|$ may not provide a good estimate of $\left\| \sum_{\alpha \in I} x_\alpha \right\|$, and, as Example 2 illustrates, an unordered sum in an infinite-dimensional Banach may converge even though it does not converge absolutely. Thus, there is no necessary and sufficient condition for the convergence of an unordered sum that involves only the norms of its terms; heuristically, the 'directions' of the terms are important in determining convergence, not just the 'lengths.' For *orthogonal* sums in a *Hilbert space*, however, there is a very simple necessary and sufficient condition for unordered convergence which we describe next.

Orthogonal unordered sums in a Hilbert space

Suppose that $\{x_\alpha \mid \alpha \in A\}$ is an indexed set of orthogonal vectors in a Hilbert space, meaning that $x_\alpha \perp x_\beta$ for $\alpha \neq \beta$. We do not assume that vectors are normalized, so $\|x_\alpha\| \ge 0$ may be any non-negative number.

13 Proposition The orthogonal unordered sum

$$\sum_{\alpha \in A} x_\alpha$$

converges in a Hilbert space $\mathcal{H}$ if and only if the non-negative unordered sum

$$\sum_{\alpha \in A} \|x_\alpha\|^2$$

converges in $\mathbb{R}$.

Proof. Suppose that I is a finite subset of A. Since $x_\alpha \perp x_\beta$ for $\alpha \neq \beta$, the Pythagorean theorem implies that

$$\left\| \sum_{\alpha \in I} x_\alpha \right\|^2 = \sum_{\alpha \in I} \|x_\alpha\|^2 .$$

Hence, from Theorem 7, the unordered sum $\sum_{\alpha \in A} x_\alpha$ is Cauchy in $\mathcal{H}$ if and only if $\sum_{\alpha \in A} \|x_\alpha\|^2$ is Cauchy in $\mathbb{R}$, which proves the result since a Hilbert space is complete. $\square$

When we combine Proposition 12 and Proposition 13, we get the following explicit criterion for the convergence of orthogonal sums.

14 Corollary An orthogonal unordered sum

$$\sum_{\alpha \in A} x_\alpha$$

converges in a Hilbert space if and only if there is a constant $0 \leq C < \infty$ such that

$$\sum_{\alpha \in I} \|x_\alpha\|^2 \leq C$$

for all finite subsets $I \subset A$.

Orthonormal bases

Suppose that $E = \{e_\alpha \mid \alpha \in A\}$ is an *orthonormal set* in a Hilbert space $\mathcal{H}$, meaning that $e_\alpha \perp e_\beta$ for $\alpha \neq \beta$ and $\|e_\alpha\| = 1$. If $x \in \mathcal{H}$ and I is a finite subset of A, then we have Bessel's inequality,

$$\sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2 \leq \|x\|^2.$$

It follows from this inequality and Corollary 14 that the orthogonal unordered sum

$$\sum_{\alpha \in A} \langle e_\alpha, x \rangle \, e_\alpha$$

converges for every $x \in \mathcal{H}$.

If the sum is $y \in \mathcal{H}$, say, then continuity of the inner-product stated in Theorem 11 and the orthonormality of E imply that

$$\langle e_\alpha, x \rangle = \langle e_\alpha, y \rangle$$

for every $\alpha \in A$, so $x - y \perp E$. If E is *complete*, meaning that the only vector orthogonal to E is 0, then $x = y$, and we can expand every $x \in \mathcal{H}$ with respect to E as the unordered sum

$$x = \sum_{\alpha \in A} \langle e_\alpha, x \rangle \, e_\alpha.$$

Furthermore, if $x, y \in \mathcal{H}$, then the continuity of the inner-product implies that

$$\langle x, y \rangle = \sum_{\alpha \in A} \overline{\langle e_\alpha, x \rangle} \, \langle e_\alpha, y \rangle \, .$$

We call a complete orthonormal set in a Hilbert space $\mathcal{H}$ an *orthonormal basis* of $\mathcal{H}$. Summarizing this discussion, we have the following theorem.

15 Theorem Suppose that $\{e_\alpha \mid \alpha \in A\}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$. Every $x \in \mathcal{H}$ is given by the unordered sum

$$x = \sum_{\alpha \in A} \langle e_\alpha, x \rangle \, e_\alpha.$$

The mapping $\iota : \mathcal{H} \to \ell^2(A)$ defined by

$$\iota(x) = \{\langle e_\alpha, x \rangle \mid \alpha \in A\}$$

is a Hilbert-space isomorphism of $\mathcal{H}$ onto the Hilbert space

$$\ell^2(A) = \left\{ \{c_\alpha\}_{\alpha \in A} \mid c_\alpha \in \mathbb{C}, \quad \sum_{\alpha \in A} |c_\alpha|^2 < \infty \right\}$$

of indexed sets of complex numbers with square-summable unordered sums.

It follows from the projection theorem and Zorn's lemma that every Hilbert space has a complete orthonormal basis, so every Hilbert space is isomorphic to $\ell^2(A)$ for some index set A. This isomorphism is not a canonical one, however, since it depends upon the choice of an orthonormal basis of $\mathcal{H}$.

One can prove (see e.g. N. Dunford and J. Schwartz, *Linear Operators*, Vol. I) that any two orthonormal bases in a Hilbert space have the same cardinality. Since elements of the Hilbert space are represented by unordered sums, only the cardinality of the orthonormal bases is important in determining the isomorphism class of a Hilbert space: If $\sigma : A \to B$ is a one-to-one, onto map, then

$$\{c_\beta \mid \beta \in B\} \mapsto \left\{c_{\sigma(\alpha)} \mid \alpha \in A\right\}$$

defines a Hilbert space isomorphism of $\ell^2(B)$ onto $\ell^2(A)$. For example, $\ell^2(\mathbb{N})$ is isomorphic to $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Q})$.

Thus, we can classify Hilbert spaces by the cardinality of their orthonormal bases. For example, any finite-dimensional (complex) Hilbert space is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, and any infinite-dimensional, separable Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.