Problem Set 1
Math 204, Spring 2004

1. Find the first two terms in an expansion as $\epsilon \to 0$ of all three roots of the cubic equation

$$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0.$$ 

(Use a dominant balance argument to determine appropriate rescalings.)

2. Find the first few terms in an expansion as $\epsilon \to 0^+$ of the solution of

$$\frac{e^{-x^2}}{x} = \epsilon.$$ 

3. Consider the matrix

$$A = \begin{pmatrix} a & \omega \\ -\omega & b \end{pmatrix},$$

where $a, b, \omega$ are real numbers and $a \neq b$. Use perturbation theory to compute the first two nonzero terms in the expansion of the eigenvalues as: (a) $\omega \to 0$; (b) $\omega \to \infty$. Show that your results agree with an expansion of the exact solution.

4. The transverse displacement $u(x, t)$ of a vibrating string with density (mass/unit length) $\rho(x) > 0$, and constant tension $T > 0$ satisfies the wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c = \sqrt{T/\rho}$ is the wave speed. If the string has length $L$ and is pinned at either end, then

$$u(0, t) = u(L, t) = 0.$$ 

An eigenmode of the string is a solution of the form

$$u(x, t) = y(x)e^{-i\omega t},$$

where $\omega$ is the frequency of the mode.

(a) Show that the frequencies $\omega$ are solutions of the eigenvalue problem

$$y'' + \frac{\omega^2}{c^2} y = 0, \quad y(0) = y(L) = 0.$$
Solve this problem when $\rho = \rho_0$ is constant.
(b) Suppose that the string is slightly nonuniform, with density

$$\rho(x) = \rho_0 [1 + \epsilon a(x)],$$

where $\epsilon \ll 1$ and $a : [0, L] \to \mathbb{R}$ is a given function. Nondimensionalize the problem with respect to $L$ and $c_0 = \sqrt{T/\rho_0}$, and compute the frequencies up to the order $\epsilon$. 