Problem Set 5  
Math 205A: Winter Quarter, 2014

1. Suppose that $\Omega \subset \mathbb{C}$ is a simply connected open set, $\gamma : [a,b] \to \Omega$ is a smooth curve in $\Omega$, and $c \in \mathbb{C} \setminus \Omega$. Show that $W_{\gamma}(c) = 0$.

2. A linear fractional transformation $f : \mathbb{C} \to \mathbb{C}$ on the extended complex plane $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ is defined for $ad - bc \neq 0$ by
   
   $$f(z) = \frac{az + b}{cz + d},$$

   with $f(\infty) = a/c$ and $f(-d/c) = \infty$ if $c \neq 0$, or $f(\infty) = \infty$ if $c = 0$.

   (a) Verify that a linear fractional transformation is a biholomorphic mapping of $\mathbb{C}$ onto itself.

   (b) Show that every biholomorphic mapping of $\mathbb{C}$ onto itself is a linear fractional transformation. (You can assume the result we proved in class that every holomorphic mapping $f : \mathbb{C} \to \mathbb{C}$ is rational.)

3. Let $\Omega$ be an open subset of $\mathbb{C}$ that is star-shaped with respect to $c \in \Omega$ (i.e., for every $z \in \Omega$, the line segment from $c$ to $z$ is included in $\Omega$).

   (a) Prove that every closed curve $\gamma : [a,b] \to \Omega \setminus \{c\}$ is homotopic to a closed curve $\beta : [a,b] \to C_r$ where $C_r = \{z \in \mathbb{C} : |z - c| = r\}$ is any circle of sufficiently small radius $r > 0$.

   (b) Show that two closed curves $\alpha, \beta : [a,b] \to C_r$ are homotopic if and only if they have the same winding numbers i.e., $W_\alpha(c) = W_\beta(c)$. Deduce that two closed curves are homotopic in $\Omega \setminus \{c\}$ if and only if their winding numbers about $c$ are equal.

   (c) If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma : [a,b] \to \Omega \setminus \{c\}$ is a closed curve, use the homotopy form of Cauchy’s theorem to prove that

   $$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - c} \, dz = W_\gamma(c) f(c).$$

4. Consider the closed curve $\gamma$ in $\Omega \setminus \{0,1\}$ shown in Figure 1.

   (a) What are the winding numbers $W_\gamma(0)$ and $W_\gamma(1)$?

   (b) Is $\gamma$ homotopic to 0 in $\Omega \setminus \{0,1\}$?

   (c) If $f$ is holomorphic in $\Omega \setminus \{0,1\}$ with poles at 0 and 1, what is $\int_{\gamma} f \, dz$?

   (You should explain your answers, but proofs are not required.)
Figure 1: Curve $\gamma$ for Problem 4.