Problem Set 3

Math 205B: Spring Quarter, 2014

1. (a) Let $0 < R(a) < \infty$ denote the radius of convergence of a power series at $a \in \mathbb{C}$, and suppose that $|b - a| < \frac{1}{2}R(a)$. Prove that the radius of convergence $R(b)$ of the power series at b of the analytic continuation of the power series at a satisfies $|R(a) - R(b)| \leq |a - b|$. (You can assume standard facts about the convergence of power series.)

(b) Deduce that if a holomorphic germ has an analytic continuation along a curve $\gamma : [0, 1] \to \mathbb{C}$, then the analytic continuation can always be obtained by analytic continuation along a finite chain of discs.

(c) Show that if $\mathbf{f}$ is the complete analytic continuation of a holomorphic germ f_a, then the collection of germs $\{f_z \in \mathbf{f}\}$ is countable for every $z \in \mathbb{C}$. *Hint.* Continue analytically by discs with rational centers.

2. (a) Consider a linear ODE

$$a(z)f'' + b(z)f' + c(z) = 0,$$

where the coefficients a, b, c are entire functions. Show that if a germ f_z satisfies this ODE at $z \in \mathbb{C}$, then every analytic continuation of f_z satisfies the ODE.

(b) Solve the ODE

$$z^2 f'' - zf' + f = 0,$$

and discuss what happens when solutions are continued analytically around the unit circle $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi i t}$.

3. Consider the multi-valued function

$$w = z^{1/3} + (z - 1)^{1/2}.$$

(a) Show that the branches of this function are solutions of an irreducible polynomial $P(z, w) = 0$ of degree 6 in w.

(b) Show that the critical values are $z = 0$, $z = 1$, $z = \infty$.

(b) Define branches $\{w_k(z) : 1 \leq k \leq 6\}$ by their values at $z = 2$:

$$w_1(2) = t + 1, \quad w_2(2) = t\lambda + 1, \quad w_3(2) = t\lambda^2 + 1,$$
$$w_4(2) = t - 1, \quad w_5(2) = t\lambda - 1, \quad w_6(2) = t\lambda^2 - 1$$

where $t = 2^{1/3} \in \mathbb{R}$ and $\lambda = e^{2\pi i/3}$. Let γ, δ be closed curves in $\mathbb{C} \setminus \{0,1\}$ with winding numbers $W_\gamma(0) = 1$, $W_\gamma(1) = 0$, and $W_\delta(0) = 0$, $W_\delta(1) = 1$. Show that analytic continuation along γ gives the permutation of branches

$$(123)(456)$$

and analytic continuation along δ gives the permutation

$$(14)(25)(36).$$

Deduce the permutation of the branches for the curve $\epsilon = \delta^{-1}\gamma^{-1}$.

(c) Draw a schematic picture of the Riemann surface of this function. How many ramification points are there and what are their orders? What is the genus of the Riemann surface?

4. For $\omega_1, \omega_2 \in \mathbb{C}^*$ with $\omega_1/\omega_2 \notin \mathbb{R}$, define the lattice

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}.$$

Let $\mathbb{C}/\Lambda$ be the corresponding Riemann surface with projection $p : \mathbb{C} \to \mathbb{C}/\Lambda$, where $p : z \mapsto [z]$. For $a \in \mathbb{C}$, define

$$a\Lambda = \{a\omega : \omega \in \Lambda\}.$$

We say that two lattices Λ, Λ' are similar if $\Lambda' = a\Lambda$ for some $a \in \mathbb{C}^*$.

(a) If Λ, Λ' are lattices with projections p, p' and $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is biholomorphic, show that there is a biholomorphic map $\tilde{f} : \mathbb{C} \to \mathbb{C}$ such that $f \circ p = p' \circ \tilde{f}$.

(b) If $a, b \in \mathbb{C}$ and $a\Lambda \subset \Lambda'$, define $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ by $f_{a,b} : [z] \mapsto [az + b]'$. Show that $f_{a,b}$ is a biholomorphism if and only if $a\Lambda = \Lambda'$, so two tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are holomorphically equivalent if and only if their lattices are similar.