# A VERY SHORT SURVEY ON BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

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Abstract. This short expository paper aims to use Dirichlet boundary value problem to elaborate on some of the interactions between complex analysis, potential theory, and harmonic analysis. In particular, I will outline Wiener's solution to the Dirichlet problem for a general planar domain using harmonic measure and prove some elementary results for Hardy spaces.

## 1. Introduction

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^2$ be an open set. A function $u \in C^2(\Omega; \mathbb{R})$ is called *harmonic* if

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } \Omega. \tag{1.1}$$

The notion of harmonic function can be generalized to any finite dimensional Euclidean space (or on (pseudo)Riemannian manifold), but the theory enjoys a qualitative difference in the planar case due to its relation to the magic properties of functions of one complex variable. Think of $\Omega \subset \mathbb{C}$ (in this paper I will use $\mathbb{R}^2$ and $\mathbb{C}$ interchangeably when there is no confusion). Then the Laplace operator takes the form

$$\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}, \text{ where } \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.2}$$

Let $\mathrm{Hol}(\Omega)$ denote the space of holomorphic functions on $\Omega$. It is easy to see that if $f \in \mathrm{Hol}(\Omega)$, then both $\Re f$ and $\Im f$ are harmonic functions. Conversely, if $u$ is harmonic on $\Omega$ and $\Omega$ is simply connected, we can construct a harmonic function $\tilde{u}$, called the **harmonic conjugate** of $u$, via Hilbert transform (or more generally, Bäcklund transform), such that $f = u + i\tilde{u} \in \mathrm{Hol}(\Omega)$. If $f \in \mathrm{Hol}(\Omega)$, then $f$ satisfies the maximal modulus principle and the Cauchy integral formula, and $f$ is complex analytic. In correspondence, it is well known from the theory of elliptic PDEs that if $u$ is harmonic, then $u$ satisfies the maximal modulus principle and the mean value property (as a counterpart of Cauchy integral formula), and $u$ is real analytic. Another important result, which mirrors Hurwitz's theorem, states that local uniform convergence of a sequence of harmonic function results in a harmonic function. This is called **Harnack's principle**. We will use this important property later in the paper.

One should also expect the qualitative difference between planar harmonic function theory and higher-dimensional harmonic function theory by the fact that the symmetry group of Laplace equation in $\mathbb{R}^n$ is the conformal group of $\mathbb{R}^n$, and the conformal groups of $\mathbb{R}^2$ and $\mathbb{R}^n$ for $n > 2$ have drastically different characterization. I will not dwell into this subject.

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*Key words and phrases.* Dirichlet problem, harmonic measure, Hardy spaces.

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I am interested in the following

**Definition 1.2.** Let $\Omega \subset \mathbb{R}^2$ be a domain (connected and open, here and below), and let $\Gamma = \partial\Omega$. The **Dirichlet problem** consists of the following: given $f \in C(\Gamma)$, find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\Delta u = 0 \quad \text{in } \Omega \tag{1.3}$$

$$u = f \quad \text{on } \Gamma \tag{1.4}$$

A domain such that the Dirichlet problem can be solved for all $f \in C(\Gamma)$ is called a **Dirichlet domain**.

A simple example of a Dirichlet domain is in place.

## 2. The unit disc and the Riemann mapping theorem

2.1. **Dirichlet problem on the unit disc.** Let $\Omega = \mathbb{D} = \{|z| < 1\}$ be the unit disc. Then $\Gamma = \mathbb{T}$ is the unit circle. It is well known that the boundary value problem

$$\Delta u = 0 \quad \text{in } \mathbb{D} \tag{2.1}$$

$$u = f \quad \text{on } \mathbb{T} \tag{2.2}$$

has a *unique* solution for each $f \in C(\mathbb{T})$. In particular, the solution has an elegant representation formula. First consider the *Poisson kernel* given by

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) \tag{2.3}$$

for $0 \leqslant r < 1$ and $\theta \in \mathbb{T}$. Note that the Poisson kernel is a harmonic approximation to the identity. The unique solution is given by convolving the boundary data with the Poisson kernel

$$u_f(r,\phi) = \frac{1}{2\pi}\int_{\mathbb{T}} f(e^{i\theta})P(r,\phi-\theta)d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} f(e^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}d\theta. \tag{2.4}$$

Standard results about convolution tells us that $u_f$ converges uniformly to $f$ as $r \to 1$.

*Remark* 2.1. The Poisson representation formula in essence owes its elegance to the $L^2$ theory on the torus and the nice form that Laplace operator takes on Euclidean domain. Also note that the integral formula (2.4) makes sense even if $f \in L^1_{loc}(\mathbb{T}) = L^1(\mathbb{T})$, so Equation 2.1 is satisfied. Since $L^1$ functions do not have pointwise value, Equation 2.2 of course does not make sense in general, but it does at points where $f$ is continuous. Moreover, in the disc model, since the convergence is uniform, we can approach any boundary point along any curve. This turns out not to be true in general. It seems like we need to synthesize a multifaceted problem:

- How does the regularity of $\Gamma$ affect solvability?
- How does the regularity of $f$ affect solvability?

In light of the Riemann mapping theorem (which I will present in the next section), I will add the third

- How does the topology of $\Omega$ affect solvability?

2.2. **Simply connected domain and boundary behavior of conformal maps.** The celebrated Riemann mapping theorem states that every simply connected proper subdomain of $\mathbb{C}$ is biholomorphic to the unit disc. This leads us to look at the boundary value problem (1.3) when $\Omega$ is a simply connected subdomain. Obviously Equation 1.3 can be solved. However, we know that conformal maps do not "see" boundaries when no conditions are put on the latter. It turns out that as long as we put slightly more constraint on $\Gamma$, then the problem is resolved. First I start with an important definition.

**Definition 2.2.** A curve $\gamma \subset \mathbb{C}$ is a **Jordan curve** if it is the image of an *injective continuous* map $\mathbb{T} \longrightarrow \mathbb{C}$. A **Jordan arc** is is the image of an injective continuous map of a closed interval into the plane. Let $\Omega \subset \mathbb{C}$ be a simply connected domain, then $\Omega$ is a **Jordan domain** if $\Gamma = \partial\Omega$ is a Jordan curve.

We have the celebrated

**Theorem 2.3.** ***(Carathéodory).*** *Let $\phi$ be a conformal mapping from $\mathbb{D}$ onto a Jordan domain $\Omega$. Then $\phi$ has a continuous extension to $\overline{\mathbb{D}}$ and the extension restricts to a homeomorphism from $\mathbb{T}$ onto $\Gamma$.*

*Remark* 2.4. The proof of the theorem is purely topological and can be found in [3].

In light of Theorem 2.3, any Jordan domain is a Dirichlet domain.

Let $\phi$ be as above. For further interest, we can investigate the relation between regularity of $\Gamma$ and that of $\phi$ on $\mathbb{T}$. We can also study the relation between regularity of $f \in C(\Gamma)$ and that of $u_f$ at points of $\Gamma$. There are two results:

**Theorem 2.5.** ***(Kellogg.)*** *Let $\phi$ be a conformal map from $\mathbb{D}$ onto a Jordan domain $\Omega$. Let $k \geqslant 1$ and $\alpha \in (0, 1)$. Then $\Gamma \in C^{k,\alpha}$, if and only if $\arg \phi' \in C^{k-1,\alpha}(\mathbb{T})$, if and only if $\phi \in C^{k,\alpha}(\overline{\mathbb{D}})$ and $\phi' \neq 0$ on $\overline{\mathbb{D}}$.*

**Corollary 2.6.** *Suppose $\Gamma \in C^{k,\alpha}$ ($k \geqslant 1$) and $f \in C^{m,\beta}(\Gamma)$. Then $u_f \in C^{n,\sigma}(\overline{\Omega})$ where $n = \min\{k, m\}$ and $\sigma = \min\{\alpha, \beta\}$ and $u_f$ is constructed in terms of conformal maps.*

This finishes the discussion on Jordan domain.

## 3. A Potential Theory Approach

In this section, I would like to experiment with some weakened boundary regularity and topological constraints and sketch the proof for Wiener's solution to the Dirichlet problem (for details, see [3]) for an arbitrary domain. The machinery is heavy, so due to the length constraint, much must be taken on faith.

Continuity does not see pathology, and the job is in turn done by measure theory.

3.1. **Motivation.** We can interpret the formula (2.4) as integrating $f$ against a measure $\omega(z, \mathbb{D})$. Let $E$ be a finite union of open arcs on $\mathbb{T}$. We define the **harmonic measure** of $E$ at $z \in \mathbb{D}$ to be

$$\omega(z, E, \mathbb{D}) = \frac{1}{2\pi}\int_E \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\theta.$$

By property of Poisson kernel and Lebesgue integral, $\omega(z, \cdot, \mathbb{D})$ extends to a Borel probability measure on $\mathbb{T}$. Our goal is to construct harmonic measure for a large class of domains.

3.2. **Finitely connected domain.** We have solved the Dirichlet problem for Jordan domain. One should expect that if the domain contains slightly more holes, namely, if $\partial\Omega$ consists of finitely many pairwise disjoint Jordan curves (in this case, we call $\Omega$ **finitely connected Jordan domain**), the result is still true. Indeed, we have

**Theorem 3.1.** *Finitely connected Jordan domains are Dirichlet.*

*Sketch of proof.* We can find two points $a, b \in \mathbb{C}\backslash\overline{\Omega}$ and a Jordan curve $\gamma$ connecting $a$ and $b$ that cuts through all the "holes" of $\Omega$ such that $\Omega\backslash\gamma$ consists of finitely many disjoint Jordan domains. We can first solve the Dirichlet problem for one connected component by specifying a consistent boundary data. By performing this procedure inductively while making sure that the boundary data is consistent, we can solve the Dirichlet problem for $\Omega$. Uniqueness is clear from maximum principle. □

Fix a finitely connected domain $\Omega$ and $z \in \Omega$. Let $u_f$ be the *unique* solution to the Dirichlet problem with boundary data $f$. Then the map $C(\partial\Omega) \longrightarrow \mathbb{R}$ defined by $f \mapsto u_f(z)$ is a bounded linear functional. By Riesz-Markov-Kakutani theorem[4], there exists a unique Borel probability measure such that

$$u_f(z) = \int_{\partial\Omega} f(\zeta) d\omega. \tag{3.1}$$

We call the quantity $\omega(E) = \omega(z, E, \Omega)$ the **harmonic measure** of the set $E \subset \partial\Omega$ at $z \in \Omega$. By uniqueness, we see that this definition agrees with the case when $\Omega = \mathbb{D}$.

3.3. **Logarithmic Capacity.**

**Definition 3.2.** Let $\nu$ be a compactly supported signed measure. The **energy integral** of $\nu$ is defined to be

$$\mathcal{E}(\nu) = \iint \log \frac{1}{|z - \zeta|} d\nu(z) d\nu(\zeta) \tag{3.2}$$

whenever the integral is absolutely convergent. Let $E \subset \mathbb{C}$ be a compact set. Denote by $\mathcal{P}_E$ the space of Borel probability measure on $E$. The **Robin's constant** of $E$ is defined to be

$$V(E) = \inf_{\nu \in \mathcal{P}_E} \mathcal{E}(\nu). \tag{3.3}$$

The **logarithmic capacity** of $E$ is defined to be $\mathrm{Cap}(E) = e^{-V(E)}$.

*Remark.* In constrast with the Lebesgue measure, which is a notion of volume, the logarithmic capacity is a notion of the ability of a set to hold a certain type of "potential energy" (the name actually comes from the term "capacitance" in physics), depending on how we define the potential energy. As an example, though the usual 1/3-Cantor set has Lebesgue measure zero, it has logarithmic capacity greater or equal to 1/9. It is not surprising that the notion of capacity should relate to Hausdorff content. An introduction can be found in Carleson's paper [5], and I will not dwell too much into the subject. Note that logarithmic capacity also plays an important role of complex dynamics, see[6].

3.4. **Statement of Wiener's result.** Let $\hat{\mathbb{C}}$ be the Riemann sphere constructed from one-point compactification. The theorem states

**Theorem 3.3.** ***(Wiener, 1924).*** *If* $\Omega \subset \hat{\mathbb{C}}$ *is such that* $\mathrm{Cap}(\mathbb{C}\backslash\Omega) > 0$. *Then* $\Omega$ *is Dirichlet.*

*Remark.* Wiener's idea is to exhaust the set $\Omega$ by finitely connected Jordan domains, and since each solution on these nice domains is associated with a harmonic measure, one can expect a harmonic measure in the limit. The hard part of the proof is the limiting argument, and I break it down into two steps.

3.5. **A technical lemma.** One major difficulty in Wiener's idea is to show that the limit does not depend on our choice of exhaustion. This is taken care of by the following

**Lemma 3.4.** *The infimum in (3.3) is uniquely attained on $\mathcal{P}_E$ if* $\mathrm{Cap}(E) > 0$.

*Discussion of proof.* Let $\Omega = \hat{\mathbb{C}} \backslash E$ be the component that includes $\infty$. If $\Omega$ is finitely connected and $\partial\Omega$ is smooth, then we can show that $\omega(\infty, \cdot, \Omega)$ is the unique minimizer. The idea is the following: first construct a Green's function[1] $g_\Omega$ associated with $\Omega$ with a pole at infinity (which is possible by Theorem 3.1), and express Robin's constant in terms of the Green's function; the regularity assumption on $\partial\Omega$ ensures that there is no boundary contribution to the area integral and helps us conclude that $\omega(\infty, \cdot, \Omega)$ is the minimizer; the positive-definiteness of $\mathcal{E}$ helps us achieve uniqueness, and during this step we have to interchange the order of integration, the validity of which is justified by the finite capacity condition.

If $E$ is any compact set, then we can consider a smooth $\infty$-exhaustion by finitely connected domain

$$\infty \in \Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1}\ ,\ \Omega = \bigcup_{n=1}^{\infty} \Omega_n\ ,\ \partial\Omega_n \in C^\infty. \tag{3.4}$$

Let $E_n = \hat{\mathbb{C}} \backslash \Omega_n$. By the previous step, $\mu_n = \omega(\infty, \cdot, \Omega_n)$ is the unique witness of the infimum of (3.3) for $E_n$. By Banach-Alaoglu theorem (or Prokhorov's theorem), we can pass onto a subsequence and obtain a weak-star limit $\mu \in \mathcal{P}_E$ (or in the language of probability, a vague limit). Since weak star convergence is lower semi-continuous, $\mu$ is a minimizer. Uniqueness follows from a variation argument. $\square$

3.6. **Sketch of Proof of Wiener's theorem.** Let $\Omega$ be smoothly exhausted as in (3.4). Continuously extend $f$ to all of $\hat{\mathbb{C}}$ in an arbitrary fashion. Let $\omega(z, \cdot, \Omega_n)$ be the harmonic measure associated with $\Omega_n$ and set

$$u_n(z) = \int_{\partial\Omega_n} f(\zeta) d\omega(z, \zeta, \Omega_n). \tag{3.5}$$

Let $E_n = \mathbb{C} \backslash \Omega_n$, $E = \mathbb{C} \backslash \Omega$, and let $u_n = \mu_{E_n}$ be defined as above, then $\mu_n \rightharpoonup^* \mu_E$. Therefore,

$$\lim_{n \to \infty} u(\infty) = \lim_{n \to \infty} \int_{\partial\Omega_n} f(\zeta) d\mu_n(\zeta) = \int_{\partial\Omega} f(\zeta) d\mu_E(\zeta). \tag{3.6}$$

The last integral obviously does not depend on the choice of exhaustion of $\Omega$ or on the choice of extension of $f$.

For $\infty \neq z \in \Omega$, we can apply a Möbius transformation $\phi$ mapping $z$ to $\infty$. Möbius transformation preserves the positivity of capacity. Therefore, $u_n(z)$ converges to a limit, which we call $u(z)$ independently of the choice of exhaustion and boundary value extension. Note that the sequence $u_n$ is uniformly bounded and harmonic, so the convergence is locally uniform. By Harnack's principle, the limit function is also harmonic and has boundary value $f$. $\square$

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[1]The modern notion of a Green's function is the fundamental solution of a differential operator, more precisely, the function is mapped to the Dirac delta by the operator in the sense of distribution.

## 4. Hardy and Dirichlet Spaces

In the previous section I mainly focus on recovering a harmonic function from its boundary data. In this section, I will investigate the "dual" of the issue, namely, given a holomorphic or harmonic function on a domain, how does it behave on the boundary? For an arbitrary domain the question is very hard to answer. In this paper, I will again return to the toy model $\Omega = \mathbb{D}$ and $\Gamma = \mathbb{T}$. I will mainly focus on the holomorphic case. The subject has been extensively studied in harmonic analysis.

4.1. **Hardy Space.** We put functions in different classes in order to understand their behaviors. For example, $L^p$-class measures the size of the function as well as their asymptotic decay behavior. However, they do not give us any information about the oscillation of the function. To compensate this, we can use the classical Sobolev space $W^{k,p}$ where $k$ is the degree of weak differentiability and it keeps track of the oscillation(frequency). The same principle carries through when we study holomorphic functions. Given $f \in \mathrm{Hol}(\mathbb{D})$, we have should have no worry of the regularity of $f$ since it is analytic. To capture the size, we consider the $p$-th moment radial maximal function

$$\mathscr{M}_p(f) = \sup_{0 \leqslant r < 1} \frac{1}{2\pi} \left( \int_{\mathbb{T}} \left| f(re^{i\theta}) \right|^p d\theta \right)^{1/p} \ , \ 0 < p < \infty \tag{4.1}$$

$$\mathscr{M}_\infty(f) = \max_{z \in \overline{\mathbb{D}}} |f(z)| = \max_{\theta \in \mathbb{T}} \left| f(e^{i\theta}) \right| . \tag{4.2}$$

and define the **Hardy space** on the unit disc as

$$H^p(\mathbb{D}) = \{ f \in \mathrm{Hol}(\mathbb{D}) : \mathscr{M}_p(f) < \infty \} \ , \ 0 < p \leqslant \infty. \tag{4.3}$$

I will suppress the domain for the rest of the paper when the context is clear. If $1 \leqslant p \leqslant \infty$, $H^p$ is a Banach space with norm $\|\cdot\|_{H^p} = \mathscr{M}_p(\cdot)$. When $0 < p < 1$, $H^p$ is a complete metric space, where the metric is again given by the maximal function raised to the $p$-th power. For expository purpose, I will focus on the case when $p = 2$ (many of the results carry through for $p \neq 2$). $H^2$ is a Hilbert space, with an inner product constructed in the usual way. To illustrate why Hardy space is an appropriate choice for studying boundary behavior, I present the following

**Theorem 4.1.** *If $f \in H^2$, then $\lim_{r \nearrow 1} f(re^{i\theta})$ exists almost everywhere on $\mathbb{T}$.*

*Proof.* Expand $f$ about the origin and we can write

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}. \tag{4.4}$$

Integrate over the circle and taking supremum, we have

$$\|f\|_{H^2} = \sup_{0 \leqslant r < 1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2 < \infty. \tag{4.5}$$

The sequence $(a_n)$ then defines an $L^2$ function on the circle, and this function is the almost everywhere limit. □

As a consequence, we can identify $H^2$ as a subspace of $L^2(\mathbb{T})$. We can denote this space by $H^2(\mathbb{T})$. Another characterization by Fourier series is immediately in place.

**Theorem 4.2.** *$f \in H^2$ if and only if $f$ is the Poisson integral of some $f(e^{i\theta}) \in L^2(\mathbb{T})$ and $\hat{f}(n) = 0$ for all $n < 0$.*

*Proof.* One direction is clear. Suppose $f \in H^2$, then let $f(e^{i\theta})$ be the radial limit as in Theorem 4.1. Convolve it against the Poisson kernel as in (2.4) produces a holomorphic function, so its power series expansion does not involve $\bar{z}$, and hence $f(e^{i\theta})$ cannot have negatively-indexed Fourier coefficient. $\square$

4.2. **Dirichlet space.** Hardy spaces, as we have seen above, only characterize the boundary behavior of holomorphic function up to size. To study smoothness, one candidate class is given by the **Dirichlet space**. To define such class, first we consider the sesquilinear form

$$\langle f, g\rangle'_{\mathcal{D}} = \int_{\mathbb{D}} f'(z)\overline{g'(z)}dA(z) \quad \text{for } f, g \in \mathrm{Hol}(\mathbb{D}) \tag{4.6}$$

where $dA(z) = \pi^{-1}dxdy$ is the normalized area measure on the disc. The form gives rise to a seminorm $[\cdot]_{\mathcal{D}}$ in the natural way. Then the Dirichlet inner product can be defined to be

$$\langle f, g\rangle_{\mathcal{D}} = \langle f, g\rangle'_{\mathcal{D}} + \langle f, g\rangle_{H^2(\mathbb{T})}. \tag{4.7}$$

This inner product gives rise to a norm. The space $\mathcal{D}$ consists of all the holomorphic functions with finite norm. Observe that if $f \in \mathcal{D}$, then $f \in H^2(\mathbb{T})$, so a Fourier characterization of the norm is immediately in place, since

$$[f]_{\mathcal{D}} = \sum_{n=1}^{\infty} n\left|\hat{f}(n)\right|^2 \ , \ \|f\|_{H^2(\mathbb{T})} = \sum_{n=0}^{\infty}\left|\hat{f}(n)\right|^2 \tag{4.8}$$

and therefore, $\|\cdot\|_{\mathcal{D}} = \mathscr{H}^{1/2}(\mathbb{T}) \cap \mathrm{Hol}(\mathbb{D})$, where $\mathscr{H}^{1/2}$ is the Sobolev space of square integrable functions with 1/2-moment integrable "derivatives" (of course, characterized by Fourier series). This new characterization exemplifies the phenomenon of loosing fractional derivatives when we take the trace of Sobolev functions.

As an important remark, the Sobolev regularity exponent 1/2 is the borderline case for one dimension, not enough for the boundary value to be continuous. However, the space $\mathscr{H}^{1/2}(\mathbb{T})$ does embed into $BMO(\mathbb{T})$, the space of functions with bounded mean oscillation. This class of functions has been extensively studied in modern harmonic analysis. To further investigate along the line, we need heavier machinery from some deep theories which ventures too far from either complex analysis or the intended scope of this project[2]. As an example, we can exploit more of the nice properties of Dirichlet spaces by considering how we can continuously embed $\mathcal{D}$ into $L^2(\mathbb{D}, \mu)$, where $\mu$ is a positive Borel measure. Of course, this is a criterion on the measure $\mu$. As it turns out, the class of measures such that this embedding is continuous (called **Carleson measure**) leads to various important results. For instance, their weak-type estimate is comparable to the Lebesgue weak-type estimate on the boundary. For more on BMO, Carleson measure, and other modern aspects of harmonic analysis, see [2] and [7].

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[2]In plain language, they have either nothing to do with the course, or I don't understand anything about them.

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