Classification and Properties of the Chebyshev Equation and its Solutions

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Background

In what follows, we study the well-known Chebyshev equation – named after the Russian mathematician Pafnuty Chebyshev (1821-1894), who made wide-ranging contributions to several areas of mathematics – from the point of view of differential equations on $\mathbb{C}$. This equation is

$$(1-z^2)w'' - zw' + x^2 w = 0$$

where x is typically a real parameter. It is a special case of a class of ODEs known as hypergeometric-type equations, which generically have the form

$$\sigma(z)\frac{d^2w}{dz^2} + \tau(z)\frac{dw}{dz} + \eta w(z) = 0$$

and also contain the Bessel equation, the Legendre equation, and many other well-known and historically significant differential equations. In fact, the Chebyshev equation is a member of a sub-class of hypergeometric type equations which are represented by the Gegenbauer equation:

$$(1-z^2)w'' - (a+b+1)w' - abw = 0.$$

It is easy to see that the Chebyshev equation as written takes this form, with $a = x$ and $b = -x$ (or vice versa). The Chebyshev equation is of particular interest in applications, because for particular choices of the parameter x its solutions generate an orthogonal sequence of polynomials, which satisfy min-max properties that make them very useful for least squares approximation.

This work is divided into (1) classification and properties of the equation itself, and (2) properties and behavior of the solutions.

Equation

We begin by analyzing the Chebyshev equation and its properties, specifically by classifying it as a regular ODE and as a hypergeometric equation and applying important known properties of these types of differential equations.

Regular Singular Points

Recall the equation

$$(1-z^2)w'' - zw' + x^2 w = 0$$

which is equivalent to

$$w'' - \frac{z}{1-z^2}w' + \frac{x^2}{1-z^2}w = 0 \tag{1}$$

for $x \in \mathbb{R}$ and $z \in \mathbb{C}$. Note that (1) is of the form

$$w'' + P(z)w' + Q(z)w = 0$$

where P and Q are meromorphic functions on $\mathbb{C}$. By Fuchs' Theorem, (1) is regular if and only if P has poles of order at most 1, and Q has poles of order at most 2. It is obvious that P and Q each have poles at $z = \pm 1$; both functions also have a pole at $z = \infty$. We claim all three of these are regular points of the ODE. To show this, we prove that the coefficients $P(z)$ and $Q(z)$ have poles of order at most 1 and at most 2 (respectively) at each of these three points, as follows.

First consider $z = 1$:

$$\begin{aligned}\lim_{z\to 1}(z-1)P(z) &= \lim_{z\to 1}\frac{z(z-1)}{(1-z)(1+z)} \\ &= -\lim_{z\to 1}\frac{z}{1+z} \\ &= -\frac{1}{2}\end{aligned}$$

so P has a pole of order 1 at $z = 1$. Similarly,

$$\begin{aligned}\lim_{z\to 1}(z-1)Q(z) &= \lim_{z\to 1}\frac{x^2(z-1)}{(1-z)(1+z)} \\ &= -\lim_{z\to 1}\frac{x^2}{1+z} \\ &= -\frac{x^2}{2}\end{aligned}$$

hence Q also has a pole of order 1 at $z = 1$.

Next we check -1 in a similar fashion:

$$\begin{aligned}\lim_{z\to -1}(z+1)P(z) &= \lim_{z\to 1}\frac{z(z+1)}{(1-z)(1+z)} \\ &= \lim_{z\to -1}\frac{z}{1-z} \\ &= \frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}
\lim_{z\to -1}(z+1)Q(z) &= \lim_{z\to -1}\frac{x^2(z+1)}{(1-z)(1+z)}\\
&= \lim_{z\to -1}\frac{x^2}{1-z}\\
&= \frac{x^2}{2}
\end{aligned}$$

so P and Q both have poles of order 1 at $z=-1$.

Finally we check the point at $z=\infty$ by making the change of coordinates

$$\xi = \frac{1}{z}$$

and evaluating the singularity at $\xi=0$ of the transformed equation. Applying the substitution to the derivative results in

$$\frac{d}{dz} = -\xi^2\frac{d}{d\xi}; \qquad \frac{d^2}{dz^2} = \xi^4\frac{d^2}{d\xi^2} + 2\xi^3\frac{d}{d\xi}.$$

If W is the transformation of the resulting solution $w(z)$, so that $W(\xi)=w(z)$, the equation in transformed coordinates is

$$\xi^4W'' + 2\xi^3W' + \frac{1/\xi}{(1-1/\xi)\,(1+1/\xi)}\xi^2W' + \frac{x^2}{(1-1/\xi)(1+1/\xi)}W = 0$$

or, simplifying,

$$\xi^4W'' + \left(2+\frac{1}{1-\xi^2}\right)\xi^3W' + \frac{\xi^2x^2}{1-\xi^2}W = 0.$$

Dividing this by ξ^4, we get

$$W'' + \left(\frac{2}{\xi}+\frac{1}{\xi(1-\xi^2)}\right)W' + \frac{x^2}{\xi^2(1-\xi^2)}W = 0.$$

We check the poles of the new coefficients at $\xi=0$:

$$\lim_{\xi\to 0}\xi\left(\frac{2}{\xi}+\frac{1}{\xi(1-\xi^2)}\right) = \lim_{\xi\to 0}(2+1) = 3$$

$$\lim_{\xi\to 0}\xi^2\frac{x^2}{\xi^2(1-\xi^2)} = x^2$$

Thus, the transformed coefficients have poles of order 1 and 2 at $z=0$, so ∞ is a Fuchsian point – that is, a regular singular point – for the original equation. We conclude that (1) is a regular ODE.

Hypergeometric Form and Consequences

We can make several observations about the behavior of (1) based on its classification as a hypergeometric type equation. To make this explicit, we make an affine change of coordinates to write the equation as the hypergeometric equation

$$z(1-z)w'' + (c - (a+b+1)z)w' - abw = 0.$$

Let $t = \frac{1}{2}(1-z)$; then $z = 1-2t$, $\frac{d}{dz} = -\frac{1}{2}\frac{d}{dt}$, and $\frac{d^2}{dz^2} = \frac{1}{4}\frac{d^2}{dt^2}$, so that the equation becomes (if $\dot{w}$ represents the derivative with respect to t)

$$t(1-t)\ddot{w} + \left(\frac{1}{2} - t\right)\dot{w} + x^2 w = 0 \tag{2}$$

which is in the form of the hypergeometric equation, where the constants are $a, b = \pm x$, and $c = \frac{1}{2}$.

Symmetry of $z = \pm 1$

In the form (2), it is easy to see that the solutions will have the same behavior near $z = 1$ and $z = -1$, by making the substitution $t \mapsto 1-t$. Under this transformation, $\frac{d}{dt} \mapsto -\frac{d}{dt}$ and $\frac{d^2}{dt^2} \mapsto \frac{d^2}{dt^2}$, and $\left(\frac{1}{2} - t\right) \mapsto \left(t - \frac{1}{2}\right)$, so that the equation remains unchanged:

$$(1-t)t\ddot{w} - \left(t - \frac{1}{2}\right)\dot{w} + x^2 w = 0$$

which implies that the indicial equation and the resulting series solutions will be identical at the two poles $z = \pm 1$ (corresponding to $t = 0$ and $t = 1$).

Monodromy Group

Furthermore, expressing the equation in the form of (2) allows us to write explicitly the matrices generating the monodromy group for this equation:

$$M(\gamma_0) = \begin{pmatrix} 1 & -1 - e^{2\pi i x} \\ 0 & -1 \end{pmatrix}$$

$$M(\gamma_1) = \begin{pmatrix} -1 & 0 \\ -1 - e^{-2\pi i x} & 1 \end{pmatrix}$$

where γ_0 and γ_1 are curves in the Riemann sphere winding around the singular points $t = 0$ (or $z = 1$) and $t = 1$ (where $z = -1$), respectively. $M(\gamma_\infty)$ satisfies

$$M(\gamma_0)M(\gamma_1)M(\gamma_\infty) = I_2$$

that is, the matrix corresponding to a curve winding around all three singularities is homotopic to the identity.

Solutions

Next, we use the indicial equation near the regular singular points of the ODE to study the solutions to (1).

Indicial Equation

To find solutions we suppose that at a pole z_j, w has the form $w = (z - z_j)^\rho$. In this case, the derivatives are $w' = \rho(z - z_j)^{\rho-1}$ and $w'' = \rho(\rho-1)(z - z_j)^{\rho-2}$. The leading order terms of the differential equation will be the $(z - z_j)^{\rho-2}$ terms. Then, we can evaluate the limit as $z \to z_j$ assuming the solution is of this form; the limit will depend on the pole z_j. For this equation, we have the following indicial equations:

At $z = 1$, the equation is

$$\rho(\rho-1)(z-1)^{\rho-2} + \frac{z}{(z-1)(z+1)}\rho(z-1)^{\rho-1} + \frac{x^2}{-(z-1)(z+1)}(z-1)^\rho = 0$$

which, to leading order as $z \to 1$, is

$$\rho(\rho-1) + \frac{1}{2}\rho = 0$$

so that

$$\rho = 0 \text{ or } \rho = \frac{1}{2}.$$

At $z = -1$, we have

$$\rho(\rho-1)(z+1)^{\rho-2} + \frac{z}{(z-1)(z+1)}\rho(z+1)^{\rho-1} + \frac{x^2}{-(z-1)(z+1)}(z+1)^\rho = 0$$

which as $z \to -1$, has the same leading order equation

$$\rho(\rho-1) + \frac{1}{2}\rho = 0$$

so that, as above,

$$\rho = 0 \text{ or } \rho = \frac{1}{2}.$$

At $z = \infty$, we use the equation for $W(\xi)$, and instead suppose that $W = \xi^m$, which results in the equation

$$m(m-1)\xi^{m-2} + \left(\frac{2}{\xi} + \frac{1}{\xi(1-\xi^2)}\right)m\xi^{m-1} + \frac{x^2}{\xi^2(1-\xi^2)}\xi^{m-2} = 0$$

or, as $\xi \to 0$ taking the $\mathcal{O}(\xi^{m-2})$ terms,

$$m(m-1) + 3m + x^2 = 0$$

so that

$$m = -1 \pm \sqrt{1-x^2}.$$

Re-substituting $\xi = \frac{1}{z}$, this results in

$$w(z) = z^{1 \pm \sqrt{1-x^2}}$$

or,

$$\rho = 1 \pm \sqrt{1-x^2}.$$

Series Solutions

Full solutions can be written in series form based on the roots of the indicial equation, specifically

$$w(z) = (z - z_j)^\rho \sum_{n=0}^{\infty} a_n z^n$$

We begin by assuming a solution of this form, and then replacing it into the differential equation to solve for the coefficients a_n at each order. This will result in recurrence relations for the coefficients.

$$\begin{aligned} w &= \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots \\ w' &= \sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \dots \\ w'' &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} = 2a_3 + 6a_3 z + 12 a_4 z^2 + \dots \end{aligned}$$

Plugging this back into (1) gives an equation for each power of z; since this holds for any z, we can find recurrence relations for the coefficients a_n by equating the powers of z.

The constant terms result in the equation

$$2a_2 + x^2 a_0 = 0 \implies a_2 = -\frac{x^2}{2} a_0$$

while the $\mathcal{O}(z)$ equation is

$$6a_3 - a_1 + x^2 a_1 = 0 \implies a_3 = \frac{1 - x^2}{6} a_1.$$

In fact, for the z^n terms in general, the equation will be

$$(n+1)(n+2)a_{n+2} - n(n-1)a_n - n a_n - x^2 a_n = 0$$

which results in the recurrence relation

$$a_{n+2} = \frac{(n-x)(n+x)}{(n+1)(n+2)} a_n.$$

The initial choice of a_0 is arbitrary; furthermore, since a_{n+2} depends only on a_n and not on a_{n+1}, a_1 can also be chosen arbitrarily. That is, the even and odd terms form two independent series. Thus, the full solution is a linear combination of the series of even terms ($n = 2k$), which is fully determined by the choice of a_0, and the series of odd terms ($n = 2k+1$), which is determined by the choice of a_1. These series are conventionally denoted $F(z)$ and $G(z)$, respectively, and explicitly they are

$$F(z) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{(x - 2(k-1))(x - 2(k-2)) \dots x^2 \dots (x + 2(k-2))(x + 2(k-1))}{(2k)!} z^{2k}$$

$$G(z) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{(x-(2k-1))(x-(2k-3))\dots(x-1)(x+1)\dots(x+(2k-3))(x+(2k-1))}{(2k+1)!} z^{2k+1}$$

So, the full solutions corresponding to $\rho = 0$ near $z = \pm 1$ are linear combinations of $F(z)$ and $G(z)$. Observe that if x is a nonnegative integer, then either F or G (depending on whether x is even or odd) terminates after $n = x$, resulting in a polynomial solution of degree x.

Next, we consider the solution corresponding to $\rho = \frac{1}{2}$ near $z = 1$. For this case, we assume solutions are of the form

$$w(z) = (z-1)^{1/2} \sum_{n=0}^{\infty} a_n z^n = (z-1)^{1/2}(a_0 + a_1 z + a_2 z^2 + \dots)$$

Rather than rearranging the series into a power series in $(z-1)$, we will use the hypergeometric (transformed) version of equation (1):

$$t(1-t)\ddot{w} + \left(\frac{1}{2} - t\right)\dot{w} + x^2 w = 0.$$

In the transformed coordinates, the $z = 1$ point becomes $t = 0$, but the indicial equation near $t = 0$ still has roots 0 and $\frac{1}{2}$. Hence, we have a series solution of the form

$$w(t) = t^{1/2} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+1/2}$$

The derivatives are

$$w'(t) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) a_n t^{n-1/2}$$
$$w''(t) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) a_n t^{n-3/2}$$

As before, we plug these back in to the ODE to obtain recurrence relations for the coefficients a_n. The first few orders areas follows:

$$\mathcal{O}(t^{-1/2}) : a_0 = a_0$$
$$\mathcal{O}(t^{1/2}) : a_1 = \frac{\left(\frac{1}{2} - x\right)\left(\frac{1}{2} + x\right)}{\frac{3}{2}} a_0$$
$$\mathcal{O}(t^{3/2}) : a_2 = \frac{\left(\frac{3}{2} - x\right)\left(\frac{3}{2} + x\right)}{5} a_1$$

In general the equation for the terms of order $\mathcal{O}(t^{n+1/2})$ is

$$-\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right) a_n + \left(n + \frac{1}{2}\right)\left(n + \frac{3}{2}\right) a_{n+1} + \frac{1}{2}\left(n + \frac{3}{2}\right) a_{n+1} - \left(n + \frac{1}{2}\right) a_n + x^2 a_n = 0$$

which simplifies to the recurrence relation

$$a_{n+1} = \frac{\left(\left(n+\frac{1}{2}\right)-x\right)\left(\left(n+\frac{1}{2}\right)+x\right)}{\left(n+\frac{3}{2}\right)(n+1)} a_n$$

so the series solution can be written

$$w(t) = a_0 \sum_{n=0}^{\infty} \frac{(n-1/2-x)\dots(1/2-x)x(x+1/2)\dots(x+n-1/2)}{\left(\frac{1}{2}\right)\dots\left(n-\frac{1}{2}\right)n\left(n+\frac{1}{2}\right)(n+1)} t^{n+1/2},$$

where a_0 is arbitrary. Now, replacing t with z as the independent variable, the solution is

$$w(t) = a_0 \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1/2}}\right) \frac{(n-1/2-x)\dots(1/2-x)x(x+1/2)\dots(x+n-1/2)}{\left(\frac{1}{2}\right)\dots\left(n-\frac{1}{2}\right)n\left(n+\frac{1}{2}\right)(n+1)} (1-z)^{n+1/2},$$

which converges for $|1-z| < 2$.

Since each solution will satisfy the differential equation, any linear combination of the above will also form a solution. So, we can write the general form of solutions near $z = \pm 1$ as

$$w(z) = a_0 F(z) + a_1 G(z) + a_2 w(z)$$

for F, G, and w as above.

Due to the symmetry of (1), the series solution at $z = 1$ will be identical. A similar method can be applied at $z = \infty$, but will depend on the parameter x; this is omitted here. However, the two solutions near ∞ are order $z^{\pm x}$ as $z \to \infty$.

The preceding method is sometimes referred to as the Frobenius method.

Contour Integrals

An alternate method of finding solutions to hypergeometric type equations is to use contour integration. Hypergeometric equations satisfy particular properties with respect to integration along certain contours; specifically, by Theorem 3.1 of ([1]), the hypergeometric equation

$$z(1-z)w'' + (c - (a+b+1)z)w' - abw = 0$$

has a solution in the form of the contour integral

$$w(z) = \int_\gamma t^{b-c}(1-t)^{c-a-1}(t-z)^{-b} dt$$

over the contour γ, if $\gamma(t) : [0,1] \to \mathbb{C}$ is any curve satisfying

$$t^{b-c+1}(1-t)^{c-a}(t-(\gamma(t)))^{-b-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

This is proven by taking derivatives of the integral and showing that with this definition for $w(z)$,

$$z(1-z)w'' + (c-(a+b+1)z)w' - abw = -b \int_\gamma \frac{\partial}{dt}\left[t^{b-c+1}(1-t)^{c-a}(t-(\gamma(t)))^{-b-1}\right] dt$$

which, by assumption on γ, must be zero.

Due to the symmetry of the Gegenbauer equation, for the Chebyshev equation this integral form of the solution is

$$w(z) = \int_\gamma (t^2-1)^{\frac{2x-1}{2}}(t-z)^{-x}dt$$

if $\gamma(t)$ satisfies

$$(t^2-1)^{\frac{2x+1}{2}}(t-\gamma(t))^{-x-1}\big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Furthermore, if $\gamma(t)$ satisfies

$$(t^2+2t\gamma(t)+1)^{x+1}t^{x-2}\big|_{\gamma(0)}^{\gamma(1)} = 0$$

then there is a solution given by

$$w(z) = \int_\gamma (t^2+2tz+1)^x t^{x-1}dt.$$

We omit the proof of the final integral representation; it relies on the symmetry of Gegenbauer equations, specifically the *Whipple transformation* based on the mapping $z \to \frac{\pm z}{\sqrt{z^2-1}}$.

Remark. It can be shown that several of the integral solutions and the series solutions are proportional to each other. This implies that the Frobenius solutions can be obtained, up to a normalization constant, by integrating certain elementary functions over contours satisfying particular conditions.

References

[1] J. Dereziski, "Hypergeometric Type Functions and Their Symmetries," 2013. [Online]. Available: http://www.fuw.edu.pl/{~}derezins/hyper-published.pdf