Hecke Operators

We recall that an *entire modular form of weight* k is an analytic function on the upper half-plane $\mathbb{H}$ that satisfies the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group Γ, and has a Fourier expansion $f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$. The fact that only non-negative powers of $e^{2\pi imz}$ occur in the expansion corresponds to the fact that f is holomorphic at infinity. We let M_k denote the set of entire modular forms of weight k and recall that M_k is a linear space over $\mathbb{C}$ and is in fact finite dimensional. We are interested in the sequence of linear operators $T_n : M_k \to M_k$ defined as follows:

$$T_n f(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right),$$

where $d|n$ is taken to imply that d is positive. While it's obvious that $T_n f$ is linear and holomorphic on $\mathbb{H}$, as a finite sum of holomorphic functions, it's not clear that $T_n f$ transforms correctly or has the right Fourier expansion. We first consider the Fourier expansion of f.

Theorem 1. If $f \in M_k$, and $f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$, then

$$T_m f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi imz},$$

where

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

In particular, we see that $T_n f$ is holomorphic at infinity.

Proof. We substitute the Fourier expansion for f into the formula for $T_n f$ and then rework the resulting expression so that it looks like a Fourier expansion. From $f(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}$, we see that

$$T_n f(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c(m) e^{2\pi im(nz+bd)/d^2}.$$

We first pull out the infinite sum, obtaining

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n} (n/d)^{k-1} c(m) e^{2\pi imnz/d^2} (1/d) \sum_{b=0}^{d-1} e^{2\pi imb/d}.$$

If d divides m, then the sum over b is a sum of 1's and so is equal to d. If d does not divide m, then by the geometric sum formula, the sum over b takes the value $[1 - e^{2\pi imb}]/[1 - e^{2\pi imb/d}] = 0$. Therefore,

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n, d|m} (n/d)^{k-1} c(m) e^{2\pi imnz/d^2}.$$

Setting $q = m/d$, we rewrite the above summation as

$$T_n f(z) = \sum_{q=0}^{\infty} \sum_{d|n} (n/d)^{k-1} c(qd) e^{2\pi i q n z/d}.$$

In the sum over d we can replace d by n/d, since each divisor of n is still included once, obtaining

$$T_n f(z) = \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c(qn/d) e^{2\pi i q d z}.$$

Now we collect, for each $m \in \mathbb{N}$, the powers $e^{2\pi i q d z}$ of $e^{2\pi i z}$ for which $qd = m$, obtaining

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n, d|m} d^{k-1} c(mn/d^2) e^{2\pi i m z}.$$

This implies the theorem, since $d|(n,m)$ iff $d|n$ and $d|m$. □

Now we check that $T_n f$ transforms correctly under Γ. It will be useful to write $T_n f$ in another form involving only one summation. By inspection one sees that

$$T_n f(z) = n^{k-1} \sum_{a \geq 1,\, ad=n,\, 0 \leq b < d} d^{-k} f\left(\frac{az+b}{d}\right).$$

If we let $Az = (az+b)/d$, then

$$T_n f(z) = \frac{1}{n} \sum_{a \geq 1,\, ad=n,\, 0 \leq b < d} a^k f(Az). \tag{1}$$

The map $z \mapsto Az$ is an example of a *transformation of order* n, namely a transformation of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where a, b, c, d are integers with $ad - bc = n$. The transformation can be represented by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ having determinant n in the obvious way provided we identify each matrix with its negative. We let $\Gamma(n)$ denote the set of all transformations of order n. Although $\Gamma(n)$ is not a group, we observe that $\Gamma(1)$ is the modular group Γ and in that case A acts as a mobius transformation. We can put an equivalence relation on $\Gamma(n)$ by calling to matrices A_1 and A_2 equivalent if they are in the same orbit of Γ under the action of left-multiplication, i.e. if $A_1 = VA_2$ for some $V \in \Gamma$. We now state two basic theorems about transformations of order n. The proofs will be omitted.

Theorem 2. A set of nonequivalent elements of $\Gamma(n)$ possessing one representative per equivalence class is given by the set of matrices of the form $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where d runs through the positive divisors of n and, for each d, $a = n/d$ and b runs through a complete residue system modulo d.

Theorem 3. If $A_1 \in \Gamma(n)$ and $V_1 \in \Gamma$, then there exist matrices $A_2 \in \Gamma(n)$ and $V_2 \in \Gamma$, such that $A_1V_1 = V_2A_2$. Moreover, if

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \quad \text{and} \quad V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

for $i = 1, 2$, then we have

$$a_1(\gamma_2 A_2 z + \delta_2) = a_2(\gamma_1 z + \delta_1).$$

Note that in the summation (1), if we replace $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with a matrix $A' = \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$, where $b \equiv b' \pmod{d}$, then the sum is unchanged, since for some integer m we have

$$f(A'z) = f\left(\frac{az + b + md}{d}\right) = f(Az + m) = f(Az),$$

since f is invariant under translations by integers. Thus we make the observation that by Theorem 2, we can write the sum in (1) defining $T_n f$ in the form

$$T_n f(z) = \frac{1}{n} \sum_A a^k f(Az), \tag{2}$$

where A runs through a complete set of nonequivalent elements in $\Gamma(n)$ of the form described in the theorem, and for each A the coefficient a^k is the k'th power of the entry a in A. We will use these results to establish the modular transformation property of $T_n f$.

Theorem 4. If $f \in M_k$, and $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, then

$$T_n f(Vz) = (\gamma z + \delta)^k (T_n f(z)).$$

Proof. Let $V \in \Gamma$ be fixed. Using the representation in (2) above, we write

$$T_n f(Vz) = \frac{1}{n} \sum_A a^k f(AVz), \tag{3}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs through a complete set of nonequivalent elements of $\Gamma(n)$ of the form in Theorem 3. By Theorems 2 and 3, for each A there exists matrices

$$A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \text{ in } \Gamma(n) \quad \text{and} \quad V' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \text{ in } \Gamma,$$

such that $AV = V'A'$ and $a(\gamma' A'z + \delta') = a'(\gamma z + \delta)$. Therefore, $a^k f(AVz) = a^k f(V'A'z)$. Since f is a modular form of weight k, we have $f(V'A'z) = (\gamma' A'z + \delta')^k f(A'z)$, so

$$a^k f(AVz) = a^k (\gamma' Az + \delta')^k f(A'z) = (a')^k (\gamma z + \delta)^k f(A'z).$$

Thus, (3) becomes

$$T_n f(Vz) = \frac{1}{n} (\gamma z + \delta)^k \sum_A (a')^k f(A'z).$$

It is easy to show that given $A, B \in \Gamma(n)$, we have $A' \sim B'$ iff $A \sim B$, so as A runs through a complete set of nonequivalent elements of $\Gamma(n)$, so does A'. Thus we have

$$T_n f(Vz) = \frac{1}{n}(\gamma z + \delta)^k \sum_{A'} (a')^k f(A'z) = (\gamma z + \delta)^k (T_n f(z)).$$

□

Corollary 5. If $f \in M_k$, then $T_n f \in M_k$ for all n. If f is a cusp form (i.e., the first term of the Fourier expansion of f is 0), then so is $T_n f$.

Proof. This follows immediately from Theorems 1 and 4. □

Before moving on we present another viewpoint on the definition of $T_n f$ that one may find more intuitive. We recall that given a modular form f of weight k, we can associate a function F on lattices $\Lambda \subset \mathbb{C}$, as follows. If $\Lambda = \mathbb{Z}.w_1 + \mathbb{Z}.w_2$, then $F(\Lambda) = w_2^{-k} f(w_1/w_2)$. First we show why F is well-defined. If $\Lambda = \Lambda'$ with $\Lambda' = \mathbb{Z}.w_1' + \mathbb{Z}.w_2'$, then there is some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. By the transformation property of f, we have

$$F(\Lambda') = (w_2')^{-k} f(w_1'/w_2') = (w_2')^{-k} f(A(w_1/w_2))$$

$$= (w_2')^{-k} f(w_1/w_2)(c(w_1/w_2) + d)^k = (w_2')^{-k} f(w_1/w_2)(cw_1 + dw_2)^k (w_2^{-k}) = w_2^{-k} f(w_1/w_2).$$

Conversely, given a function on lattices $\Lambda \to F(\Lambda)$ which transforms by $F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$, for $0 \neq \lambda \in \mathbb{C}$, we can associate a function f on $\mathbb{H}$ by $f(z) = F(\mathbb{Z}.z + \mathbb{Z}.1)$, which is a modular form if f is holomorphic on $\mathbb{H}$ and at infinity. We define a sequence of transformations T_n on M_k as follows. If $f \in M_k$, and F is the corresponding function on lattices indicated above, define $T_n f$ by

$$T_n F(\Lambda) = \sum F(\Lambda'),$$

where the sum ranges over all sublattices $\Lambda' \subset \Lambda$ of index n. Any such sublattice Λ' is obtained by applying an elements $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma(n)$ to the basis $(z, 1)$ of Λ. Thus

$$\begin{aligned} F(\Lambda') &= F(\mathbb{Z}.(az + b) + \mathbb{Z}.(cz + d)) \\ &= (cz + d)^{-k} F(\mathbb{Z}.\left(\frac{az + b}{cz + d}\right) + \mathbb{Z}.1) \\ &= (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right). \end{aligned}$$

Furthermore one can check that if $A \sim A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in the sense defined above (same orbit of Γ), then the identity

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = (c'z + d')^{-k} f\left(\frac{a'z + b'}{c'z + d'}\right)$$

follows from the modularity of f. Thus, letting $\Gamma\backslash\Gamma(n)$ denote the equivalence classes, we have

$$T_n f(z) = \sum_{A\in\Gamma\backslash\Gamma(n)} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

If we multiply by the normalizing constant n^{k-1} and choose representatives in $\Gamma\backslash\Gamma(n)$ of the upper-triangular form in Theorem 2 (so $c=0$), then we obtain

$$T_n f(z) = n^{k-1}\sum_A d^{-k} f(Az) = \frac{1}{n}\sum_A a^k f(Az),$$

which agrees with (3) above.

For example, consider the simple case of $T_n f$ for $n=2$. According to our first definition of T_n, we have

$$T_2 f(z) = 2^{k-1}\sum_{d|2} d^{-k}\sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right) = 2^{k-1}f(2z) + \frac{1}{2}f\left(\frac{z}{2}\right) + \frac{1}{2}f\left(\frac{z+1}{2}\right). \tag{4}$$

Now we note that $\mathbb{Z}.z+\mathbb{Z}.1$ has three sublattices of index two, namely those corresponding to the bases $(2z,1),(z,2)$, and $(z+1,2)$. Hence we have

$$\begin{aligned} 2^{k-1}\sum F(\Lambda') =& 2^{k-1}\left[F(\mathbb{Z}.2z+\mathbb{Z}.1) + F(\mathbb{Z}.z+\mathbb{Z}.2) + F(\mathbb{Z}.(z+1)+\mathbb{Z}.2)\right] \\ =& 2^{k-1}\left[f(2z) + 2^{-k}f\left(\frac{z}{2}\right) + 2^{-k}f\left(\frac{z+1}{2}\right)\right], \end{aligned}$$

which agrees with (4).

Another interesting property of the Hecke operators is that they commute.

Theorem 6. For any two Hecke operators T_n and T_m defined on M_k, we have the composition formula

$$T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{mn/d^2}.$$

We omit the proof, but observe that as a consequence of the theorem T_n and T_m commute since the right-hand side is symmetric in m and n.

As an interesting application of the preceeding theory we can prove a famous result of Ramanujan. By Corollary 5, $T_n f$ is a cusp form whenever f is a cusp form. Recall that the space $M_{12,0}$ of cusp forms of weight 12 is 1-dimensional, spanned by the discriminant Δ, which implies that Δ is an eigenfunction of T_n for all n. (Such a function is called a *simultaneous eigenform.*) The Fourier expansion of Δ is given by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z},$$

where τ is the Ramanujan function, with $\tau(1)=1$. By Theorem 1, the Fourier expansion of $T_n\Delta$ begins with $\tau(n)e^{2\pi i z}$..., so that the eigenvalue of Δ for T_n is $\tau(n)$. In other words, $T_n\Delta = \tau(n)\Delta$ for all n. This implies that $\gamma_n(m) = \tau(n)\tau(m)$ holds for all m and n, where as in Theorem 1 $\gamma_n(m)$ is the m'th Fourier coefficient of $T_n\Delta$. Recalling the definition of $\gamma_n(m)$, we obtain

$$\tau(m)\tau(n) = \sum_{d|(n,m)} d^{11}\tau\left(\frac{mn}{d^2}\right).$$

To conclude we remark that Petersson discovered that for all k, the vector space $M_{k,0}$ has a basis of simultaneous eigenforms. The outline of the proof is to introduce an inner product on the (finite-dimensional) space of cusp forms with respect to which the Hecke operators are self-adjoint. Since they also commute by Theorem 6, it follows from linear algebra they can be simultaneously diagonalized. The result can be extended to show that M_k has a basis of simultaneous eigenforms for all k.

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