Hecke Operators

We recall that an entire modular form of weight $k$ is an analytic function on the upper half-plane $\mathbb{H}$ that satisfies the transformation property

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group $\Gamma$, and has a Fourier expansion $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi imz}$. The fact that only non-negative powers of $e^{2\pi imz}$ occur in the expansion corresponds to the fact that $f$ is holomorphic at infinity. We let $M_k$ denote the set of entire modular forms of weight $k$ and recall that $M_k$ is a linear space over $\mathbb{C}$ and is in fact finite dimensional. We are interested in the sequence of linear operators $T_n : M_k \rightarrow M_k$ defined as follows:

$$T_n f(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f \left( \frac{nz + bd}{d^2} \right),$$

where $d|n$ is taken to imply that $d$ is positive. While it’s obvious that $T_n f$ is linear and holomorphic on $\mathbb{H}$, as a finite sum of holomorphic functions, it’s not clear that $T_n f$ transforms correctly or has the right Fourier expansion. We first consider the Fourier expansion of $f$.

**Theorem 1.** If $f \in M_k$, and $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi imz}$, then

$$T_m f(z) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi imz},$$

where

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} e^{\frac{mn}{d^2}}.$$

In particular, we see that $T_n f$ is holomorphic at infinity.

**Proof.** We substitute the Fourier expansion for $f$ into the formula for $T_n f$ and then rework the resulting expression so that it looks like a Fourier expansion. From $f(z) = \sum_{m=0}^{\infty} c(m) e^{2\pi imz}$, we see that

$$T_n f(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c(m) e^{2\pi im(nz + bd)/d^2}.$$

We first pull out the infinite sum, obtaining

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n} (n/d)^{k-1} c(m) e^{2\pi imz/d^2} (1/d) \sum_{b=0}^{d-1} e^{2\pi imb/d}.$$  

If $d$ divides $m$, then the sum over $b$ is a sum of 1’s and so is equal to $d$. If $d$ does not divide $m$, then by the geometric sum formula, the sum over $b$ takes the value $[1 - e^{2\pi imb}] / [1 - e^{2\pi imb/d}] = 0$. Therefore,

$$T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n,d|m} (n/d)^{k-1} c(m) e^{2\pi imz/d^2}. $$
Setting \( q = m/d \), we rewrite the above summation as

\[
T_n f(z) = \sum_{q=0}^{\infty} \sum_{d|n} (n/d)^{k-1} c(qd) e^{2\pi i q n z/d}.
\]

In the sum over \( d \) we can replace \( d \) by \( n/d \), since each divisor of \( n \) is still included once, obtaining

\[
T_n f(z) = \sum_{q=0}^{\infty} \sum_{d|n} d^{k-1} c(qn/d) e^{2\pi i q d z}.
\]

Now we collect, for each \( m \in \mathbb{N} \), the powers \( e^{2\pi i q d z} \) of \( e^{2\pi i z} \) for which \( qd = m \), obtaining

\[
T_n f(z) = \sum_{m=0}^{\infty} \sum_{d|n,d|m} d^{k-1} c(mn/d^2) e^{2\pi i m z}.
\]

This implies the theorem, since \( d|\gcd(n,m) \) iff \( d|n \) and \( d|m \).

Now we check that \( T_n f \) transforms correctly under \( \Gamma \). It will be useful to write \( T_n f \) in another form involving only one summation. By inspection one sees that

\[
T_n f(z) = \frac{1}{n} \sum_{a \geq 1, a d = n, 0 \leq b < d} a^{k-1} f\left(\frac{az + b}{d}\right).
\]

If we let \( A z = (az + b)/d \), then

\[
T_n f(z) = \frac{1}{n} \sum_{a \geq 1, a d = n, 0 \leq b < d} a^{k} f(A z).
\]

The map \( z \mapsto A z \) is an example of a transformation of order \( n \), namely a transformation of the form

\[
z \mapsto \frac{az + b}{cz + d},
\]

where \( a, b, c, d \) are integers with \( ad - bc = n \). The transformation can be represented by a matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

having determinant \( n \) in the obvious way provided we identify each matrix with its negative. We let \( \Gamma(n) \) denote the set of all transformations of order \( n \). Although \( \Gamma(n) \) is not a group, we observe that \( \Gamma(1) \) is the modular group \( \Gamma \) and in that case \( A \) acts as a mobius transformation. We can put an equivalence relation on \( \Gamma(n) \) by calling to matrices \( A_1 \) and \( A_2 \) equivalent if they are in the same orbit of \( \Gamma \) under the action of left-multiplication, i.e. if \( A_1 = V A_2 \) for some \( V \in \Gamma \). We now state two basic theorems about transformations of order \( n \). The proofs will be omitted.

**Theorem 2.** A set of nonequivalent elements of \( \Gamma(n) \) possessing one representative per equivalence class is given by the set of matrices of the form \( A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \), where \( d \) runs through the positive divisors of \( n \) and, for each \( d \), \( a = n/d \) and \( b \) runs through a complete residue system modulo \( d \).
Theorem 3. If \( A_1 \in \Gamma(n) \) and \( V_1 \in \Gamma \), then there exist matrices \( A_2 \in \Gamma(n) \) and \( V_2 \in \Gamma \), such that \( A_1 V_1 = V_2 A_2 \). Moreover, if

\[
A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix} \quad \text{and} \quad V_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}
\]

for \( i = 1, 2 \), then we have

\[
a_1(\gamma_2 A_2 z + \delta_2) = a_2(\gamma_1 z + \delta_1).
\]

Note that in the summation (1), if we replace \( A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) with a matrix \( A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \), where \( b \equiv b' \pmod{d} \), then the sum is unchanged, since for some integer \( m \) we have

\[
f(A'z) = f\left(\frac{az + b + md}{d}\right) = f(Az + m) = f(Az),
\]

since \( f \) is invariant under translations by integers. Thus we make the observation that by Theorem 2, we can write the sum in (1) defining \( T_n f \) in the form

\[
T_n f(z) = \frac{1}{n} \sum_A a^k f(Az),
\]

(2)

where \( A \) runs through a complete set of nonequivalent elements in \( \Gamma(n) \) of the form described in the theorem, and for each \( A \) the coefficient \( a^k \) is the \( k \)'th power of the entry \( a \) in \( A \). We will use these results to establish the modular transformation property of \( T_n f \).

Theorem 4. If \( f \in M_k \), and \( V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \), then

\[
T_n f(Vz) = (\gamma z + \delta)^k(T_n f(z)).
\]

Proof. Let \( V \in \Gamma \) be fixed. Using the representation in (2) above, we write

\[
T_n f(Vz) = \frac{1}{n} \sum_A a^k f(AVz),
\]

(3)

where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) runs through a complete set of nonequivalent elements of \( \Gamma(n) \) of the form in Theorem 3. By Theorems 2 and 3, for each \( A \) there exists matrices

\[
A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \quad \text{in} \quad \Gamma(n) \quad \text{and} \quad V' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \quad \text{in} \quad \Gamma,
\]

such that \( AV = V'A' \) and \( a(\gamma' A' z + \delta') = a'(\gamma z + \delta) \). Therefore, \( a^k f(AVz) = a^k f(V'A'z) \). Since \( f \) is a modular form of weight \( k \), we have

\[
f(V'A'z) = (\gamma' A' z + \delta')^k f(A'z),
\]

so

\[
a^k f(AVz) = a^k(\gamma' A z + \delta')^k f(A'z) = (a')^k(\gamma z + \delta)^k f(A'z).
\]

Thus, (3) becomes

\[
T_n f(Vz) = \frac{1}{n} (\gamma z + \delta)^k \sum_A (a')^k f(A'z).
\]
It is easy to show that given $A, B \in \Gamma(n)$, we have $A' \sim B'$ iff $A \sim B$, so as $A$ runs through a complete set of nonequivalent elements of $\Gamma(n)$, so does $A'$. Thus we have

$$T_n f(Vz) = \frac{1}{n} (\gamma z + \delta)^k \sum_{A'} (a')^k f(A'z) = (\gamma z + \delta)^k (T_n f(z)).$$

**Corollary 5.** If $f \in M_k$, then $T_n f \in M_k$ for all $n$. If $f$ is a cusp form (i.e., the first term of the Fourier expansion of $f$ is 0), then so is $T_n f$.

**Proof.** This follows immediately from Theorems 1 and 4. 

Before moving on we present another viewpoint on the definition of $T_n f$ that one may find more intuitive. We recall that given a modular form $f$ of weight $k$, we can associate a function $F$ on lattices $\Lambda \subset \mathbb{C}$, as follows. If $\Lambda = \mathbb{Z}.w_1 + \mathbb{Z}.w_2$, then $F(\Lambda) = w_2^{-k} f(w_1/w_2)$. First we show why $F$ is well-defined. If $\Lambda = \Lambda'$ with $\Lambda' = \mathbb{Z}.w'_1 + \mathbb{Z}.w'_2$, then there is some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$ 

By the transformation property of $f$, we have

$$F(\Lambda') = (w'_2)^{-k} f(w'_1/w'_2) = (w'_2)^{-k} f(A(w_1/w_2))$$

$$= (w'_2)^{-k} f(w_1/w_2)(c(w_1/w_2) + d) = (w'_2)^{-k} f(w_1/w_2)(cw_1 + dw_2)^k (w_2^{-k}) = w_2^{-k} f(w_1/w_2).$$

Conversely, given a function on lattices $\Lambda \to F(\Lambda)$ which transforms by $F(\lambda \Lambda) = \lambda^{-k} F(\Lambda)$, for $0 \neq \lambda \in \mathbb{C}$, we can associate a function $f$ on $\mathbb{H}$ by $f(z) = F(\mathbb{Z}.z + \mathbb{Z}.1)$, which is a modular form if $f$ is holomorphic on $\mathbb{H}$ and at infinity. We define a sequence of transformations $T_n$ on $M_k$ as follows. If $f \in M_k$, and $F$ is the corresponding function on lattices indicated above, define $T_n f$ by

$$T_n F(\Lambda) = \sum F(\Lambda'),$$

where the sum ranges over all sublattices $\Lambda' \subset \Lambda$ of index $n$. Any such sublattice $\Lambda'$ is obtained by applying an elements $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma(n)$ to the basis $(z, 1)$ of $\Lambda$. Thus

$$F(\Lambda') = F(\mathbb{Z}.(az + b) + \mathbb{Z}.cz + d)$$

$$= (cz + d)^{-k} F(\mathbb{Z}. \frac{az + b}{cz + d} + \mathbb{Z}.1)$$

$$= (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right).$$

Furthermore one can check that if $A \sim A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ in the sense defined above (same orbit of $\Gamma$), then the identity

$$(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) = (c'z + d')^{-k} f \left( \frac{a'z + b'}{c'z + d'} \right).$$
follows from the modularity of $f$. Thus, letting $\Gamma \backslash \Gamma(n)$ denote the equivalence classes, we have

$$T_n f(z) = \sum_{A \in \Gamma \backslash \Gamma(n)} (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right).$$

If we multiply by the normalizing constant $n^{k-1}$ and choose representatives in $\Gamma \backslash \Gamma(n)$ of the upper-triangular form in Theorem 2 (so $c = 0$), then we obtain

$$T_n f(z) = n^{k-1} \sum_A d^{-k} f(Az) = \frac{1}{n} \sum_A d^k f(Az),$$

which agrees with (3) above.

For example, consider the simple case of $T_n f$ for $n = 2$. According to our first definition of $T_n$, we have

$$T_2 f(z) = 2^{k-1} \sum_{d|2} d^{-k} \sum_{b=0}^{d-1} f \left( \frac{nz + bd}{d^2} \right) = 2^{k-1} f(2z) + \frac{1}{2} f \left( \frac{z}{2} \right) + \frac{1}{2} f \left( \frac{z+1}{2} \right).$$

(4)

Now we note that $\mathbb{Z}.z + \mathbb{Z}.1$ has three sublattices of index two, namely those corresponding to the bases $(2z, 1)$, $(z, 2)$, and $(z + 1, 2)$. Hence we have

$$2^{k-1} \sum A' = 2^{k-1} \sum A (2) + 2^{k-1} \sum A (z) + 2^{k-1} \sum A (z + 1),$$

which agrees with (4).

Another interesting property of the Hecke operators is that they commute.

**Theorem 6.** For any two Hecke operators $T_n$ and $T_m$ defined on $M_k$, we have the composition formula

$$T_m T_n = \sum_{d|\text{lcm}(m,n)} d^{k-1} T_{mn/d^2}.$$ 

We omit the proof, but observe that as a consequence of the theorem $T_n$ and $T_m$ commute since the right-hand side is symmetric in $m$ and $n$.

As an interesting application of the preceding theory we can prove a famous result of Ramanujan. By Corollary 5, $T_n f$ is a cusp form whenever $f$ is a cusp form. Recall that the space $M_{12,0}$ of cusp forms of weight 12 is 1-dimensional, spanned by the discriminant $\Delta$, which implies that $\Delta$ is an eigenfunction of $T_n$ for all $n$. (Such a function is called a simultaneous eigenform.) The Fourier expansion of $\Delta$ is given by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i nz},$$

where $\tau$ is the Ramanujan function, with $\tau(1) = 1$. By Theorem 1, the Fourier expansion of $T_n \Delta$ begins with $\tau(n) e^{2\pi i nz} \ldots$, so that the eigenvalue of $\Delta$ for $T_n$ is $\tau(n)$. In other words, $T_n \Delta = \tau(n) \Delta$ for all $n$. This implies that $\gamma_n(m) = \tau(n) \tau(m)$ holds for all $m$ and $n$, where as in Theorem 1 $\gamma_n(m)$ is the $m$'th Fourier coefficient of $T_n \Delta$. Recalling the definition of $\gamma_n(m)$, we obtain

$$\tau(m) \tau(n) = \sum_{d|\text{lcm}(m,n)} d^{11} \tau \left( \frac{mn}{d^2} \right).$$
To conclude we remark that Petersson discovered that for all $k$, the vector space $M_{k,0}$ has a basis of simultaneous eigenforms. The outline of the proof is to introduce an inner product on the (finite-dimensional) space of cusp forms with respect to which the Hecke operators are self-adjoint. Since they also commute by Theorem 6, it follows from linear algebra they can be simultaneously diagonalized. The result can be extended to show that $M_k$ has a basis of simultaneous eigenforms for all $k$.

REFERENCES
