First fundamental Theorem of Nevanlinna Theory

MAT 205B

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The theory aims to describe the value distribution of meromorphic functions by looking at various formulae connecting the values of meromorphic functions with the distribution of its zeros and poles. In the beginning we will derive some of these formulae and later use them to get some connections with main "characteristics" of meromorphic functions. In the next section we will describe some preliminary results which will prove useful in our later endeavors.

1 Preliminary Results

Let D be a bounded region with boundary Γ consisting of piecewise analytic curves, $\frac{\partial}{\partial n}$ denote the differentiation along inwardly directed normal vector to Γ, and Δ be the 2-D Laplacian operator. The following formula is called the second Green's formula

$$\int\int_D (u\Delta v - v\Delta u)d\sigma = -\int_\Gamma \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dl \tag{1}$$

The above formula can be derived from the regular Green's formula by applying it to $(-u\frac{\partial v}{\partial y}, u\frac{\partial v}{\partial x})$

$$\int\int_D \left(u\Delta v + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)d\sigma = \int_\Gamma -u\frac{\partial v}{\partial n}dl$$

Reversing u and v and subtracting we get the second Green's formula. Next we need the concept of Green's function. On a domain D the Green's function is a function $G(\zeta, z)$ defined for $\zeta, z \in \overline{D}, \zeta \neq z$, satisfying the following

1. For each $z \in D$
   $$G(\zeta, z) = -\ln|\zeta - z| + h_z(\zeta)$$
   where h_z is a harmonic function in D and continuous in $\overline{D}$.

2. If $\zeta \in \Gamma, z \in \overline{D}$ or vice-versa
   $$G(\zeta, z) = 0$$

The Green's function is unique which can be seen by considering two Green's functions G and H. Using the properties of Green's function the function $v(\zeta) = G(\zeta, z) - H(\zeta, z)$ is harmonic and continuous in $\bar{D}$. Also it is zero on the boundary of D. By the maximum and minimum modulus principle of harmonic functions, v has to be zero everywhere. Hence G=H. Using the max and min principle we can also show that Green's function is greater than 0 for $\zeta, z \in D$.

For simply connected domains the existence can be proved by using Riemann mapping theorem. This gives us a conformal function from D onto unit disc $w(\zeta) = f_z(\zeta)$ such that $f_z(z) = 0$. This function is continuous on $\bar{D}$ and has modulus 1 on the boundary. Then the Green's function is given by

$$G(\zeta, z) = \ln\frac{1}{|f_z(\zeta)|} \tag{2}$$

Example 1.1.

1. For $D = \{|\zeta| < R\}$. Then

$$G(\zeta, z) = \ln\left|\frac{R^2 - \zeta\bar{z}}{R(\zeta - z)}\right| \tag{3}$$

2. For $D = \{|\zeta| < R, Im\zeta > 0\}$. Then

$$G(\zeta, z) = \ln\left|\frac{R^2 - \zeta\bar{z}}{R(\zeta - z)}\frac{R(\zeta - \bar{z}}{R^2 - \zeta z}\right| \tag{4}$$

Suppose D be a simply connected region. Let $\gamma_1, \gamma_2, ..., \gamma_p$ be the analytic curves forming Γ. Let A_l, l=1,2,...,p be the common end points of these curves, let $\pi\alpha_l(0 < \alpha_l < 2)$ be the angles between γ_l and $\gamma_{l+1}(\gamma_{p+1} = \gamma_1)$. Then f_z has an extension to a domain containing $\bar{D}\backslash\{A_1, ...A_p\}$. We wil assume that in a sufficiently small neighborhood U_l of A_l, f_z has a representation

$$f_z(\zeta) = (\zeta - A_l)^{\frac{1}{\alpha_l}}\phi_l(\zeta) + w_l \tag{5}$$

with ϕ_l analytic in U_l, $\phi_l(A_l) \neq 0$ and $|w_l| = 1$. For our purpose it is enough to see that that this is true for the two domains in example (1.1).
Since $G(\zeta, z) > 0$ for $\zeta, z \in D$ and G is zero on the boundary, $\frac{\partial G}{\partial n} > 0$. Along the curve we have

$$\frac{f_z'(\zeta)}{f_z(\zeta}d\zeta = i\frac{\partial G}{\partial n}dl(\zeta \neq A_l) \tag{6}$$

This can be proved by taking the logarithmic derivative of f_z and using the fact that $|f_z(\zeta)| = 1$ on the boundary. Then using Cauchy Riemann equations we can get the above equation.

Theorem 1.2. Let D be a simply connected domain with a piecewise analytic boundary Γ, and let u(z) be a twice continuously differentiable function in some domain containing $\bar{D}$, excluding a finite set of points $\{c_1, c_2, ...c_q\} \subset \bar{D}$. In a neighborhood of these points u has the form

$$u(z) = d_k \ln|z - c_k| + u_k(z) \tag{7}$$

where d_k are constants and u_k is a twice continuously differentiable function in a neighborhood of the point c_k. Then

$$u(z) + \frac{1}{2\pi}\int\int_D G(\zeta, z)\Delta u(\zeta)d\sigma = \frac{1}{2\pi}\int_\Gamma u(\zeta)\frac{\partial G}{\partial n}ds - \sum_{c_k \in D} d_k G(c_k, z) \tag{8}$$

Proof. Let us exclude from D the discs of radius ϵ centered at $c_1, ..., c_q, z, A_1, ..A_p$. We construct the discs such that there is no overlap between the discs and if the center of the discs is in D then the whole disc is in D. Let the domain obtained be D_ϵ and the part of Γ not in any of the discs by Γ_ϵ. By C(ϵ, a) we denote the intersection of $\{z : |z - a| = \epsilon\}$ and D.

Letting u=$u(\zeta)$ and v=$G(\zeta,z)$ in the second Green's formula and observing that $\Delta v = 0$ we get

$$\int\int_{D_\epsilon} v\Delta u d\sigma = \left(\int_{\Gamma_\epsilon} + \int_{C(\epsilon,z)} + \sum_{k=1}^{q}\int_{C(\epsilon,c_k)} + \sum_{l=1}^{p}\int_{C(\epsilon,A_l)}\right)\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}ds\right) \tag{9}$$

If any of A_l matches with c_k we consider it only once. Let us find the limits as $\epsilon \to 0$. Since v=0 on Γ we have

$$\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = \int_{\Gamma} u(\zeta)\frac{\partial G}{\partial n}$$

By using mean value theorem we have for $a \in \bar{D}$

$$\int_{C(\epsilon,a)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}ds\right) = (\text{length of } \mathrm{C}(\epsilon,a))\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}ds\right)\Big|_{\zeta*}$$

$\zeta*$ being a point on $C(\epsilon,a)$. Length of $C(\epsilon,z)$ is $2\pi\epsilon$ and we have the following estimates as $\epsilon \to 0$

$$v = \ln\tfrac{1}{\epsilon} + O(1), \tfrac{\partial v}{\partial n} = -\tfrac{1}{\epsilon} + O(1), u = u(z) + o(1), \tfrac{\partial u}{\partial n} = O(1)$$

then,

$$\lim_{\epsilon\to o}\int_{C(\epsilon,a)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = -2\pi u(z) \tag{10}$$

If $c_k \in D$, then on $C(\epsilon,c_k)$, we have the following estimates

$$u = d_k\ln\epsilon + O(1), \tfrac{\partial u}{\partial n} = \tfrac{d_k}{\epsilon} + O(1),$$
$$v = G(c_k,z) + o(1), \tfrac{\partial v}{\partial n} = O(1)$$

Therefore for $c_k \in D$ we have

$$\lim_{\epsilon\to o}\int_{C(\epsilon,c_k)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = -2\pi d_k G(c_k,z)$$

If a is one of the points c_k belonging to the boundary or one of the points $A_1,..A_p$, the length of $(C(\epsilon,a) < 2\pi\epsilon$ and the following estimates are valid

$$u = O(|\ln\epsilon|), \tfrac{\partial u}{\partial n} = O(\tfrac{1}{\epsilon}), v = o(1), \tfrac{\partial v}{\partial n} = O(\tfrac{1}{\sqrt{\epsilon}})$$

The last of these comes from equation 5. Hence we have

$$\lim_{\epsilon\to o}\int_{C(\epsilon,c_k)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = 0$$

Now taking limit $\epsilon \to 0$ in equation 9 we get the statement of the theorem. □

2 Poisson Jensen formulae

In this and the next sections we will prove a bunch of formulae connecting the behavior of meromorphic function to their zeros and poles. Most of these are special cases of 8.

Theorem 2.1. Let D be a simply connected region with a piecewise analytic boundary Γ and $f(z) \not\equiv 0$ meromorphic function in $\bar{D}$. Then

$$\ln|f(z)| = \frac{1}{2\pi}\int_{\Gamma} \ln|f(\zeta)|\frac{\partial G}{\partial n}dl - \sum_{a_m \in D} G(a_m, z) + \sum_{b_n \in D} G(b_n, z) \tag{11}$$

where a_m and b_n are the zeros and poles of f(z) respectively.

Proof. We apply Theorem 1.2 with u(z)=ln $|f(z)|$. If z is neither a pole nor a zero then Δu is zero. Hence the integral on the right is zero. If c_k is a zero(pole) of f(z) of order χ_k then $d_k = \chi_k (d_k = -\chi_k)$. Hence the right hand side of equation 8 is equal to the right hand side of equation 11 □

One of the most important special case of the above case is when D is the disc $\{z : |z| < R\}$.

Theorem 2.2. Let $f(z) \not\equiv 0$ be meromorphic on the disc $\{z : |z| < R\}$. Then the following formula known as Poisson Jensen formula holds

$$\ln|f(z)| = \frac{1}{2\pi}\int_{0}^{2\pi} \ln|f(Re^{i\theta})| Re \frac{Re^{\imath\theta+z}}{Re^{i}\theta - z}d\theta - \sum_{|a_m|<R} \ln\left|\frac{R^2 - \overline{a_m}z}{R(z-a_m)}\right| + \sum_{|b_n|<R} \ln\left|\frac{R^2 - \overline{b_n}z}{R(z-b_n)}\right| \tag{12}$$

where a_m and b_n are the zeros and poles of f(z) respectively.

Proof. The theorem follows from entering the explicit expression for the Green's function in Theorem 2.1 and using equation 6 to get $\frac{\partial G}{\partial n}$. □

Theorem 2.3. Let $f(z) \not\equiv 0$ be meromorphic on the disc $\{z : |z| < R\}$ and let

$$f(z) = c_\lambda z^\lambda + c_{\lambda+1}z^{\lambda+1} + ..., c_\lambda \neq 0 \tag{13}$$

be the laurent expansion. Then the following known as the Jensen formula holds

$$\ln|c_\lambda| = \frac{1}{2\pi}\int_{0}^{2\pi} \ln|f(Re^{i\theta})| d\theta - \sum_{0<|a_m|<R} \ln\frac{R}{|a_m|} + \sum_{0<|b_n|<R} \ln\frac{R}{|b_n|} - \lambda \ln R \tag{14}$$

where a_m and b_n are the zeros and poles of f(z) respectively.

Proof. Let $z \to 0$ in the Poisson Jensen formula. For the sum on the right we get

$$\begin{aligned} -\sum_{|a_m|<R} + \sum_{|b_n|<R} &= -\sum_{0<|a_m|<R} + \sum_{0<|b_n|<R} + \lambda \ln\frac{|z|}{R} \\ &= -\sum_{0<|a_m|<R} \ln\frac{R}{a_m} + \sum_{0<|b_n|<R} \ln\frac{R}{b_n} + \lambda\ln\frac{|z|}{R} + o(1) \end{aligned}$$

and on the left hand side we get

$$\ln|f(z)| = \lambda \ln|z| + \ln|c_\lambda| + o(1) \tag{15}$$

Now taking $z \to 0$ the Jensen formula follows. □

3 Shimizu Ahlfors formula

This is again a special case of Theorem 1.2.

Theorem 3.1. Let $f(z) \not\equiv 0$ be meromorphic on the disc $\{z : |z| < R\}$. If $f(0)\neg\infty$, then the following formula known as Shimizu Ahlfors formula holds

$$\begin{aligned}\frac{1}{\pi}\int\int_{|z|\leq R}\left(\ln\frac{R}{|z|}\right)\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}d\sigma &= \frac{1}{2\pi}\int_0^{2\pi}\ln\sqrt{1+|f(Re^{i\theta}|^2}d\theta \\ &\quad -\ln\sqrt{1+|f(0)|^2}+\sum_{|b_n|<R}\ln\frac{R}{|b_n|}\end{aligned} \tag{16}$$

If f(0)=∞, then the right hand is replace by

$$\frac{1}{2\pi}\int_0^{2\pi}\ln\sqrt{1+|f(Re^{i\theta}|^2}d\theta - \ln|c_\lambda| - \lambda\ln R + \sum_{|b_n|<R}\ln\frac{R}{|b_n|}$$

here c_λ is the same as in Theorem 2.3.

Proof. We use Theorem 1.2 with $u(z) = \ln\sqrt{1+|f(z)|^2}$, D the disc with radius R. Let A=Re$f(z)$ and B=Im$f(z)$. We compute $\Delta u(z)$. Using the Cauchy Riemann equation and the fact that A and B are harmonic we find

$$\Delta u = \frac{2|f'(z)|^2}{(1+|f(z)|^2)^2}$$

Observing that in a neighborhood of a pole b_k of f of order χ_k, c_k in Theorem 1.2 is b_k and $d_k = -\chi_k$, we get

$$\begin{aligned}&\ln\sqrt{1+|f(z)|^2}+\frac{1}{\pi}\int\int_{|\zeta|\leq R}\ln\left|\frac{R^2-\zeta z}{R(z-\zeta)}\right|\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}d\sigma(\zeta) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}\ln\sqrt{1+|f(Re^{i\theta})|^2}d\theta + \sum_{|b_n|<R}\ln\left|\frac{R^2-\overline{b_n}z}{R(z-b_n)}\right|\end{aligned} \tag{17}$$

If f(0)$\neq\infty$ taking z to 0 we get the formula. In case f(0)$=\infty$ we take into account

$$\ln\sqrt{1+|f(z)|^2} = \lambda\ln\frac{|z|}{R} + \ln|c_\lambda| + o(1)$$

$$\sum_{|b_n|<R}\ln\left|\frac{R^2-\overline{b_n}z}{R(z-b_n)}\right| = \lambda\ln\frac{|z|}{R} + \sum_{0<|b_n|<R}\ln\frac{R}{b_n} + o(1)$$

when $z\to 0$. Taking $z\to 0$ we get the formulae. □

4 Nevanlinna Characteristics. The first fundamental theorem

We will define several real functions defined on $[0,\infty)$ which characterize the behavior of a meromorphic function $f(z)$. These functions are called the Nevanlinna characteristics of $f(z)$. The number of poles of $f(z)$ in the disc $\{|z| \le R\}$ is denoted by n(r,f). It is an integer valued function, non decreasing and right semi continuous. Any point r, is a point of discontinuity if f has poles on the circle of radius r. The jump value is equal to the number of poles on the circle. Since the poles of f can't have limit point, $n(r,f)$ is piecewise constant on every $[a,b] \subset [0,\infty)$.
Let

$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t} dt + n(0,f)\ln r \tag{18}$$

This characteristic describes the location of poles of $f(z)$.The function $N(r,f)$ is continuous, non decreasing on $(0,\infty)$. Also, as a function of $\ln r$, this is a convex function. It is noteworthy that the characteristic takes into account only the absolute value of the poles. The Jensen formula can be succinctly written in terms of N(r,f) in the following way

$$N\left(r,\frac{1}{f}\right) - N(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|d\theta - \ln|c_\lambda| \tag{19}$$

This can be seen from the following equalities

$$\sum_{0<|b_n|<r} \ln\frac{r}{|b_n|} = N(r,f) - n(0,f)\ln r$$
$$\sum_{0<|a_m|<r} \ln\frac{r}{|a_m|} = N\left(r,\frac{1}{f}\right) - n\left(0,\frac{1}{f}\right)\ln r$$
$$n(0,f) - n\left(0,\frac{1}{f}\right) = -\lambda$$

Let us now define the function $\ln^+ x$ which is useful in defining another characteristic.

$$\ln^+ x = max(\ln x, 0)$$

We have the following relations for this function

$$\ln x = \ln^+ x - \ln^+\frac{1}{x}, |\ln x| = \ln^+ x + \ln^+\frac{1}{x}, \ln^+ x = \ln x^* \tag{20}$$

where $x \ge 0$,$x^* = max(x,1)$. We will also require the following inequalities

$$\ln^+ \left| \prod_1^n x_m \right| \le \sum_1^n \ln^+ |x_m| \tag{21}$$

$$\ln^+ \left| \sum_1^n x_m \right| \le \sum_1^n \ln^+ |x_m| + \ln n \tag{22}$$

$$\left| \ln^+ |x_1| - ln^+ |x_2| \right| \le \left| \ln \left| \frac{x_1}{x_2} \right| \right| \tag{23}$$

$$\left| \ln^+ |x_1| - \ln^+ |x_2| \right| \le \ln^+ |x_1 - x_2| + \ln 2 \tag{24}$$

Let

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta$$

This function gives us information about the growth of the function $f(z)$.When $a \neq \infty$ the function $m\left(r, \frac{1}{f-a}\right)$ tells us about the proximity of $f(z)$ to a on the circle $\{|z| = R\}$ and the function $n\left(r, \frac{1}{f-a}\right)$ characterize the modulii of the roots of the equation $f(z) = a$. When the function f we are dealing with is clear, we write $n(f, a)$, $m(f, a)$ and so on instead of $n\left(r, \frac{1}{f-a}\right)$, $m\left(r, \frac{1}{f-a}\right)$. Introduce a new characteristic

$$T(r, f) = m(r, f) + N(r, f) \tag{25}$$

Now we are in a position to prove the first fundamental theorem of Nevanlinna theory.

Theorem 4.1. Let $f(z)$ be a non constant meromorphic function. Then if $a \neq \infty$

$$m(r, a) + N(r, a) = T(r, f) + \epsilon(r, a) \tag{26}$$

where $\epsilon(r, a) = O(1)$ as $r \to \infty$

Proof. We use the form of Jensen formula as given in equation 19. We can do this since the function $f - a$ is non constant. Since the poles of $f - a$ coincide with the poles of f we have

$$N\left(r, \frac{1}{f-a}\right) - N(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}) - a| d\theta - \ln |c_\lambda(a)| \tag{27}$$

where $c_\lambda(a)$ is the first non zero term in the Laurent expansion of $f - a$ in the neighborhood of 0. By the first equality in 20 we have

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}) - a| d\theta = m(r, f - a) - m\left(r, \frac{1}{f-a}\right) \tag{28}$$

and we can write equation 27 as.

$$N(r,a) + m(r,a) = N(r,f) + m(r,f-a) - \ln|c_\lambda(a)| \tag{29}$$

Now we use the inequality 24 to get

$$|\ln^+|f-a| - \ln^+|f|| \leq \ln^+|a| + \ln 2$$
$$|m(r,f-a) - m(r,f)| \leq \ln^+|a| + \ln 2$$

Defining

$$\epsilon(r,a) = m(r,f-a) - m(r,f) - \ln|c_\lambda(a)|$$

we get equation 26. To complete the proof observe that

$$|\epsilon(r,a)| \leq \ln^+|a| + \ln 2 + |\ln|c_\lambda(a)|| = O(1) \text{ as } r \to \infty$$

□

Intuitively the theorem states that if $f(z)$ takes on the value a more often than it takes the value b, then it approaches a more slowly than it approaches b. Now we show that the $T(r,f)$ tends to ∞ as $r \to \infty$, so the contribution of the term $\epsilon(r,a)$ can be ignored. Let $a = f(0)$, then $n(0,a) > 0$ and

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a)\ln r \geq n(0,a)\ln r \to \infty$$

since $m(r,a)$ is non negative

$$T(r,f) \geq N(r,a) + O(1) \to \infty \text{ as } r \to \infty \tag{30}$$

5 Further topics in Nevanlinna theory

Writing the theorem as first fundamental theorem surely means that we have a second fundamental theorem. The second theorem deals with $\overline{N}(r,f)$ defined in the same way as $N(r,f)$ but without taking multiplicity of poles into account and gives bound on $\sum_1^k m(r,a_i,f)$ where a_i are distinct values on the Riemann sphere. One of the application of the second fundamental theorem is to prove the Picard's theorem.

An interesting topic to read parallel to Nevanlinna theory is the Ahlfors theory which is the geometric conterpart of Nevanlinna theory. Lars Ahlfors was awarded the Fields medal for his work on this in 1936.

References

[1] Anatoly A. Goldberg, Iossif V. Ostrovskii. Value distribution of Meromorphic Functions, Translations of Mathematical Monographs Volume 236.

[2] W.K Hayman. Meromorphic Functions

[3] Nevanlinna Theory,
https://en.wikipedia.org/wiki/Nevanlinna_theory