Plurisubharmonic Functions and Pseudoconvex Domains

Thomas Jackson

June 8, 2018

1 Introduction

The purpose of this project is to give a brief background of basic complex analysis in several complex variables by setting sights on defining and characterizing plurisubharmonic functions and pseudoconvex domains; it will start with the basic definitions and point out some general features before moving on to describe these objects and some of their basic properties. It will end with a few brief pointers toward how they figure in some larger results in several complex variables, and will provide some references for further reading.

Plurisubharmonic functions were first introduced in a foundational 1906 paper of Hartogs, though he gave them no name; he used them to describe pseudoconvex domains, which were later proved to be equivalent to domains of holomorphy with the work of Kiyoshi Oka; this is known as the Levi Problem, which Oka proved for $\mathbb{C}^2$ in 1942, and proved it for complex dimension $n \geq 2$ in 1954. We introduce the basic concepts required to understand the mechanics of plurisubharmonic functions and the domains they characterize.

2 Basic Functional Notions

We start with the basic notions of functions of several complex variables, which are taken as analogous to the definitions from single-variable complex analysis.

We start with two definitions of a holomorphic function of n complex variables, with n a positive integer. We use the convention that for some point $z = (z_1, ..., z_n) \in \mathbb{C}^n$, the norm

$$|z|_{\mathbb{C}^n} = \sqrt{\sum_{i=1}^{n} |z_i|^2_{\mathbb{C}}}.$$

Throughout $|z|$ will be understood to be the complex norm for the appropriate dimension of z.

Definition 2.1. A function f of a point $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ is *holomorphic* at z_0 if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in some neighborhood of z_0; that is, f is holomorphic if it has a convergent power series representation in some neighborhood of z_0.

Following by analogy from the single variable case, we can define a holomorphic function by the existence of its naturally defined derivatives.

Definition 2.2. A function f of a point $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ is *holomorphic* at a point z_0 if all of its partial derivatives $\frac{\partial f}{\partial z_i}$ exist at z_0 and are finite.

It is a non-trivial result of Hartogs that these two definitions of holomorphic are equivalent in arbitrary finite dimension n, and we assume this result. There are a few concise proofs of this in several of the references at the end. [3]

For general order derivatives we want to use a convenient multi-index notation, defined in the following way

Definition 2.3. If f is a function of n variables, then for any $a = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$ we have

$$\frac{\partial^a f(z)}{\partial z^a} = \frac{\partial^{a_1}}{\partial {z_1}^{a_1}} ... \frac{\partial^{a_n}}{\partial {z_n}^{a_n}} f(z)$$

From this definition, as in the single variable case, we have that sums, differences, and products of holomorphic functions are holomorphic, and we have that quotients of holomorphic functions are holomorphic so long as the function in the numerator is non-zero. The proof of these assertions follows straightforwardly from the proofs of the single-variable case.

One nice property we've already seen in class that carries over from the single variable case is the generalized form of Cauchy's formula, which holds in multiple dimensions. We state it for clarity.

Statement 1. If f is a holomorphic function on a connected subset G of C^n then for any point $z \in G$ we have

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial G} \frac{f(w)}{z - w} dw$$

where

$$\frac{1}{w - z} = \prod_{i=1}^{n} \frac{1}{w_i - z_i}$$

and $dw = dw_1 ... dw_n$

Before jumping into functions and their domains we should familiarize ourselves with some of the basic sets. We'd also like to define some familiar sets in $\mathbb{C}^n$, namely the polydisc and the unit ball.

Definition 2.4. The open unit ball centered at a point $a \in \mathbb{C}^n$ of radius r is the set

$$B^n(a,r) = \{z : |z| < r\}$$

Here we note that the norm is the $\mathbb{C}^n$-norm; if we were to bound the set by individually bounding the $\mathbb{C}$-norm on each variable separately, we get the polydisk.

Definition 2.5. The open polydisc centered at a point $a \in \mathbb{C}^n$ of radius r is the set

$$D^n(a,r) = \{z : |z_i - a_i| < r\}$$

An important fundamental result due to Poincaré is that the open polydisc and the unit ball of dimension $n \geq 2$ are not biholomorphic. This result underscores a common theme of several complex variables: that generalizing our definitions from single variable complex analysis does not lead to structures with the same useful properties in several variables; moreover how we generalize these structures furnishes the resultant object with vastly different properties. From the perspective the single variable case, one of the most useful and profound results is the Riemann mapping theorem; however from Poincaré's result we have that there is no higher-dimensional analogue of this. So in this sense several complex variables are more tricky to deal with setwise. [2] [1]

However the happy compromise of several variables is that having more complexity of the space provides more restrictions on "well-behaved" functions. A good example of this is the Hartogs extension theorem, a form of which was proved in class.

There are many structures in several variables which are trivial in the single variable case. One of these concepts is a domain of holomorphy, which we define next.

Definition 2.6. A domain of holomorphy is an open subset U of $\mathbb{C}^n$ such that there exists a function f which is holomorphic on U but for which there does not exist an open subset $U \subset V$ on which f can be analytically continued

Intuitively, a domain of holomorphy is a set for which there is a holomorphic function that is only holomorphic on that set and cannot be analytically continued on to a larger set. In the case of a single complex variable it turns out any domain in the complex plane is a domain of holomorphy, although the proof of this is outside the scope of this project. The upshot of the Hartogs extension theorem is that there exist domains in higher dimensions which are not domains of holomorphy. We provide the following example for the two variable case, and prove it without referring to the Hartogs extension theorem.

Theorem 1. The open set $U = \{z \in \mathbb{C}^2 : 1/2 < |z| < 1\}$ is not a domain of holomorphy.

Proof. We show that any holomorphic function in $\mathbb{C}^2$ that is holomorphic on this domain is also holomorphic on the unit disc. First we define

$$\phi(z) = \frac{1}{2\pi i}\int_{|w|=3/4} \frac{f(w,z_2)}{w - z_1} dw.$$

We note that this function is holomorphic on the set

$$V = \{(z_1, z_2) : |z_1| < \frac{3}{4}, |z_2| < \frac{\sqrt{7}}{4}\}$$

and we also have by Cauchy's theorem and the fundamental theorem of Hartogs that $f(z) = \phi(z)$ on the set

$$W = \{(z_1, z_2) : |z_1| \leq \frac{3}{4}, \frac{1}{2} < |z_2| < \frac{\sqrt{7}}{4}\}$$

and since $W \subset U$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|w|=3/4} \frac{f(w, z_2)}{w - z_1} dw$$

and so we also have f is holomorphic in W and since $V \cap U$ is a domain we have ϕ is an analytic continuation of f onto the unit disk, so we have that U is not a domain of holomorphy. □

Many of the useful properties of domains of holomorphy become more apparent when their relationship to less intuitively defined sets is examined. We delve into some functional analysis in several dimensions in order to provide a glimpse of this.

3 Toward Plurisubharmonicity

3.1 Harmonic and Subharmonic functions

We define upper-semicontinuous, harmonic, subharmonic, and plurisubharmonic functions, though the reader probably has some familiarity with the first two concepts already.

We recall the definition of a harmonic function and an upper-semicontinuous function.

Definition 3.1. A twice-differentiable function f in n variables is harmonic on the domain U if

$$\frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} + ... + \frac{\partial^2 f}{\partial z_n^2} = 0$$

We remember that harmonicity is a real notion, and requires some tinkering to get it to make sense in a complex setting. The most common way of doing this is describing a function as a function of z and $\overline{z}$.

Definition 3.2. A function f is upper-semicontinuous on a set U if for any point $x_0 \in U$ we have $f(x_0) = \lim_{x \to x_0} \sup f(x)$ where the supremum is taken over all open sets containing x_0.

Upper-semicontinuous functions are really useful when working with inequalities that need to be preserved without needing the rigid structure of continuity.

Definition 3.3. A function $f(z, \overline{z}) = f(x, y)$ is subharmonic on U if it satisfies the following three conditions:

(1) $-\infty \leq f < \infty$ on U;

(2) f is upper-semicontinuous on U;

(3) for any arbitrary subdomain $U' \subset U$ and any arbitrary function g which is harmonic in U' and continuous in $\overline{U'}$, then $f(z) \leq g(z)$ on $\partial U'$ implies $f(z) \leq g(z)$ on U'.

Subharmonic functions have many useful properties; for instance linear combinations with positive coefficients of subharmonic functions are subharmonic. We also have a useful monotonic convergence theorem, where the limit of any monotonically decreasing sequence of subharmonic functions is subharmonic. The following theorem gives an equivalent formulation of subharmonicity and reveals some of their useful properties

3.1.1 A necessary and sufficient condition for subharmonicity

Theorem 2. Let $P(z, \zeta)$ be the Poisson kernel, with

$$P(z, \zeta) = \frac{1}{2\pi} \frac{r^2 - \rho^2}{r^2 - 2r\rho cos(\phi - \theta) + \rho^2}$$

where $z = re^{i\theta}$ and $\zeta = \rho e^{i\phi}$. Then we have the following equivalence. Let $f(z) < \infty$ be a subharmonic function on the ball $B^n(z_0, r)$, and suppose f is upper-semicontinuous on $\overline{B^n(z_0, r)}$; then for all $z \in B^n(z_0, r)$,

$$f(z) \leq \int_0^{2\pi} P(z - z_0, re^{i\theta}) f(z_0 + re^{i\theta}) d\theta$$

Moreover, if $f(z)$ is upper-semicontinuous and $f(z) < \infty$ and f satisfies the inequality

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

for all $z \in U$, then we have that f is subharmonic on U.

Proof. Let $f(z) < \infty$ be subharmonic in the ball $B^n(z_0, r)$ and upper-semicontinuous on $\overline{B^n(z_0, r)}$. Then it follows from the upper-semicontinuity that there is a decreasing sequence of continuous functions $f_a(z)$ that converges to $f(z)$ on $B^n(z_0, r)$. Let $g_a(z)$ be a harmonic function on $B^n(0, r)$ that assumes the value $f_a(z)$ on $\partial B^n(0, r)$. so we have

$$g_{a+1}(z) = f_{a+1}(z) \leq g_a(z) = f_a(z) \forall z \in \partial B^n(0, r)$$

We recall the maximum principle in n dimensions, which allows us to know that

$$g_{a+1}(z) \leq g_a(z) \forall z \in \overline{B^n(z_0)}.$$

We assume that a decreasing sequence of harmonic functions converges uniformly either to $-\infty$ or to a harmonic function. Then we see that $F(z) =$

$\lim_{a\to\infty} f_a(z)$ is a harmonic function or is identically $-\infty$. By the fact that f is subharmonic, we have

$$f(z) \leq g_a(z)$$

on the boundary and so

$$f(z) \leq g_a(z) \forall z \in B^n(0,r)$$

this gives us

$$f(z) \leq F(z).$$

Since each $g_a(z)$ is harmonic it is known that we have

$$\begin{aligned} F(z) &= \lim_{a\to\infty} \frac{1}{2\pi}\int_0^{2\pi} P(z-z_0, re^{i\theta}) f_a(z_0 + re^{i\theta}) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{a\to\infty} P(z-z_0, re^{i\theta}) f_a(z_0 + re^{i\theta}) d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} P(z-z_0, re^{i\theta}) F(z_0 + re^{i\theta}) d\theta. \end{aligned} \tag{1}$$

so we have the forward direction.

The reverse is a more merciful proof by contradiction. Let $f(z) < \infty$ be upper-semicontinuous and satisfy the inequality

$$f(z) \leq \frac{1}{2\pi}\int_0^{2\pi} P(z-z_0, re^{i\theta}) f(z_0 + re^{i\theta}) d\theta$$

Moreover, let $F(z)$ be a harmonic function on K compactly contained in the domain U, which is continuous on ∂K and satisfies $f(z) \leq F(z)$ on ∂K; suppose for contradiction that there exists a point $z_1 \in K$ such that $F(z_1) > f(z_1)$. We note that the function $g(z) = F(z) - f(z)$ is upper-semicontinuous on K and $g(z) \leq 0$ on ∂K but $g(z_1) > 0$. We know that it must obtain a maximum on K by virtue of the upper-semicontinuity, and say it achieves this maximum $M > 0$ for some point $z_2 \in K$. Now since $F(z)$ is harmonic and $f(z)$ satisfies the assumed inequality, we have that $g(z)$ also satisfies the first inequality. However, this is impossible, as this would demand that $F(z) \geq f(z)$ on the boundary. Thus we have that the assumed inequality holds and $f(z)$ is subharmonic. □

We can think of this abstractly as being a looser condition than harmonicity; many useful inequalities and preservation of relations follow from this fact. Many of these properties can also be generalized to plurisubharmonic functions, which are important in defining pseudoconvex domains, so we define them next.

Definition 3.4. A function f is plurisubharmonic on a domain U if it satisfies the following two criteria:

(1) f is upper-semicontinuous on U.

(2) for any arbitrary $z_0 \in U$ and some $z_1 \in U$ determined by z_0 we have that $f(z_0 + \lambda z_1)$ is subharmonic with respect to λ.

This extends subharmonicity to not only the domain in question some complex lines passing through points in the domain. We note that as in the case of subharmonicity, plurisubharmonicity is preserved under limits superior. At first glance this seems a somewhat arbitrary set of properties; we supply the reader with a quick test for plurisubharmonicity which generalizes from the case of subharmonicity; this test hopefully acts to prime the reader on their applications.

Theorem 3. A Necessary and Sufficient Condition for Plurisubharmonicity Let $P(z, \zeta)$ be the Poisson kernel. Then we have the following equivalence: Let $f(z) < \infty$ be a plurisubharmonic function on the set U, and suppose the set $U' = \{z' : z + \lambda a\}$ is compactly contained in U, then

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + rae^{i\theta}) d\theta$$

for all $z \in U$. Similarly the converse also holds.

Proof. We note that this follows directly from the definition of plurisubharmonicity and the condition for subharmonicity of $f(z_0 + az_1)$ □

We provide some examples of plurisubharmonic functions. For f a holomorphic function on U, we have that $\log(|f(z)|)$ and $|f(z)|^p$ for $p \geq 0$ are all plurisubharmonic. If $u(z)$ is a plurisubharmonic function, then so is $e^{u(z)}$ and $u^p(z)$ for $p \geq 0$. One of the most useful properties of plurisubharmonic functions which we refer to in passing is that plurisubharmonic functions are mapped to plurisubharmonic functions under biholomorphic mappings.

4 Pseudoconvex Domains

We introduce the notion of pseudoconvex domains; an important connection exists between convexity and domains of holomorphy. We define convexity and pseudoconvexity next.

Definition 4.1. A nonempty set U is convex if for any two points z_1 and z_2 we have that for all $0 \leq \lambda \leq 1$ we have $\lambda z_1 + (1 - \lambda) z_2 \in U$

Definition 4.2. A nonempty set U is logarithmically convex if for any two points z_1 and z_2 we have that for all $0 \leq \lambda \leq 1$ we have $\lambda \log(|z_1|) + (1 - \lambda) \log(|z_2|) \in U$

It is a non-trivial though not particularly difficult result that every logarithmically convex domain is pseudoconvex. In general, pseudoconvex domains are an important tool in the study of domains of holomorphy; however they require some upbuilding from more elementary notions in functional analysis.

Definition 4.3. Let $U \subset \mathbb{C}^n$ be a domain, and let $z \in U$. Let $d_U(z)$ be the function which gives the distance from the point z to the boundary ∂U. We say that the domain U is *pseudoconvex* if $d_U(z)$ is a plurisubharmonic function

This definition implies an interesting consequence: for any pseudoconvex domain, the function

$$g(z) = max\{-\log(d_U(z)), |z|^2\}$$

is plurisubharmonic on U and approaches ∞ on the boundary ∂U. We prove the following useful facts about pseudoconvex domains.

Theorem 4. An arbitrary component of the interior of the intersection of pseudoconvex domains is a pseudoconvex domain

Proof. Let U_a denote a set of pseudoconvex domains. Then we have that $-\log(d_{U_a}(z))$ are plurisubharmonic functions on U_a. Let U be any component of $\bigcap_a U_a$. Then the set of functions $-\log(d_{U_a}(z))$ are uniformly bounded locally, since the distance to the boundary on any arbitrary U_a is necessarily larger than the distance to the boundary for any compactly contained subset of U_a. This implies that

$$-\log(d_U(z)) = \sup_a[-\log(d_{U_a}(z))]$$

and more importantly, since the lim sup of plurisubharmonic functions is plurisubharmonic, we have U is pseudoconvex since its distance function is plurisubharmonic. □

Theorem 5. The union of an increasing sequence of pseudoconvex domains is pseudoconvex

Proof. Let $U_a \subset U_{a+1}$ and $U = \bigcup_a U_a$. Then

$$-\log(d_{U_a}(z)) \geq -\log(d_{U_{a+1}}(z)) \to -\log(d_U(z))$$

as $a \to \infty$ in any arbitrary subdomain $V \subset U$. Moreover the functions $\log(d_{U_a}(z))$ are plurisubharmonic on V and so therefore $-\log(d_U(z))$ is also; since V is arbitrary we have that $-\log(d_U(z))$ is plurisubharmonic on U and so U is pseudoconvex. □

From these facts we see some useful properties of pseudoconvex domains. One of the most useful characterizations of pseudoconvex domains is by the weak continuity principle, which we outline.

Definition 4.4. Given a set U we characterize the weak continuity principle as follows. Let $V_a \subset U$ be domains such that they and their boundaries lie in a set of two-dimensional analytic curves $\{C_a\}$, and let $V_a \cup \partial V_a$ be compactly contained in U and let

$$\lim_{a\to\infty} V_a = V$$

and let

$$\lim_{a\to\infty} \partial V_a = W$$

be compactly contained in U. Then we say U obeys the weak continuity principle if it follows that if V is bounded, then V is compactly contained in U.

Again this is a definition which is only interesting in the case of several complex variables, as this principle holds for any domain of $\mathbb{C}$. As it turns out, the weak continuity principle is a necessary and sufficient condition for pseudoconvexity, though we won't prove this fact here. Furthermore this fact is used to prove that all domains of holomorphy are pseudoconvex; this was first proved by Hartogs, and the converse was proved later by Oka. We provide a few references for the reader that treat of this problem. [5] [4]

References

[1] Daniel Huybrechts. *Complex Geometry.*

[2] Oliver Knill. A short introduction to several complex variables.

[3] Jiri Lebl. Tasty bits of several complex variables.

[4] Kiyoshi Oka. *The Collected Papers.*

[5] Vasiliy Vladimirov. *Methods of the Theory of Functions of Many Complex Variables.*