Quasiconformal Maps and Circle Packings

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1 Introduction

Recall the statement of the Riemann mapping theorem:

Theorem 1 (Riemann Mapping). If R is a simply connected region in the plane that is not all of the plane, then there exists a conformal equivalence $f : R \to \mathbb{D}$ where $\mathbb{D}$ is the open unit disk.

We refer to f as *the Riemann mapping* for R. We know that f exists but there seems to be no way to construct it. Thurston conjectured that given any region R, the Riemann mapping f could be approximated by mappings between circle packings. In this paper, we outline how this approximation works and give an outline of the theory behind the proof. The proof relies on the theory of *quasiconformal maps* which we introduce in the next section. Necessary background on circle packing is outlined in Section 3.

Our presentation follows parts of [4] as well as [1].

2 Quasiconformal Mappings

A *conformal map* f of the complex plane is an invertible holomorphic map between open subsets $U, V \subset \mathbb{C}$. It can be shown (see [4]) that the inverse of f must also be holomorphic.

Conformal maps can alternatively be though of as maps which preserve angles and shapes locally. Quasiconformal mappings generalize conformal mappings by allowing for some amount of deviation from the strict preservation of angles.

In order to give a formal definition of quasiconformality we need a augmentation of the Riemann mapping theorem known as Caratheodory's theorem as well as the notion of *Jordan quadrilaterals*

2.1 Caratheodory's theorem and Jordan quadrilaterals

A *Jordan curve* is a continuous loop in the plane with no self-intersections.

Recall the Riemann mapping theorem which states that a simply connected subset U of the plane is conformally equivalent to the unit disk $\mathbb{D}$. Caratheodory's theorem extends this conformal equivalence to the closed unit disk in the special case when U is bounded by a Jordan curve.

Theorem 2 (**Caratheodory's theorem**)**.** If f is a conformal equivalence $\mathbb{D} \to U$ in $\mathbb{C}$ then f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{U}$ if and only if δU is a Jordan curve.

Definition 2.1. A Jordan quadrilateral (Q, q_1, q_2, q_3, q_4) is an open region Q enclosed by a Jordan curve along with four distinct points q_1, q_2, q_3, q_4 on the Jordan curve enclosing Q. The points q_i are located in counterclockwise order and are called the vertices of the quadrilateral.

Using the Riemann mapping theorem and Caratheodory's theorem, one can prove the following (see [2])

Proposition 2.1. Let (Q, q_1, q_2, q_3, q_4) be a Jordan quadrilateral. There is a unique (up to affine transformation) rectangle $R = [0, a] \times [0, b]$ and conformal map $h : Q \to R$ which maps vertices to vertices, meaning

$$h(q_1) = 0 \qquad h(q_2) = a \qquad h(q_3) = a + bi \qquad h(q_4) = ib.$$

The value a/b is called the conformal modulus (or modulus) of Q and is denoted $\mathrm{mod}(Q)$.

Before defining quasiconformal maps, we give an intuitive explanation. Recall that conformal maps are homeomorphisms which locally map very small circles to circles. Quasiconformal maps take small circles to small "distorted" circles. The extent to which a quasiconformal map can distort circles is bounded by some fixed parameter; K in the below definition. Note also that our definitions use rectangles rather than circles. This is only because they make the definition simpler.

Definition 2.2. Assume $K \geq 1$ and let $\phi : U \to V$ be an orientation-preserving homeomorphism of open $U, V \subset \mathbb{C}$. We say that ϕ is *K-quasiconformal* on U if

$$\mathrm{mod}(\phi(Q)) \leq K \mathrm{mod}(Q)$$

for every Jordan quadrilateral Q in U.

Quasiconformal maps are of course closely related to conformal maps. If the "distortion factor" K is 1 there is no distortion. To make this precise: a homeomorphism is 1-quasiconformal iff it is conformal. [4]

We defined the conformal modulus of a quadrilateral above. In fact, an analogue of the conformal modulus can be defined for annuli in place of quadrilaterals. Without getting into the details, the conformal modulus $\mathrm{mod}(A)$ of an annulus is defined as

$$\inf\{\|\rho\|_{L^2} : \ f \text{ is Borel measurable and } \int_\gamma \rho(z)|dz| \geq 1\}$$

where γ is any degree 1 curve in A. We will not use this definition. We only need the following two facts:

Fact 1: The modulus of an annulus $\{z : r < |z - z_0| < R\}$ is $\log \frac{R}{r}$.

Fact 2: The conformal modulus is a conformal invariant. Furthermore, if ϕ is K-quasiconformal, then $\mathrm{mod}(\phi(A)) \leq K \mathrm{mod}(A)$.

2.2 Basic results on quasiconformal maps

We need two results on quasiconformal maps. Proofs are given in [4].

Recall that we say a sequence ϕ_n of mappings $\phi_n : U \to V$ converges *locally uniformly* to ϕ if every point of U has a neighborhood on which ϕ_n converges uniformly to ϕ. The following result shows that in certain cases, quasiconformality is preserved in the limit.

Theorem 3. Let $K \geq 1$ and $\phi_n : U \to V_n$ a sequence of K-quasiconformal maps converging locally uniformly to a homeomorphism $\phi : U \to V$. Then ϕ is also K-quasiconformal.

Another result we need is that K-quasiconformal maps can be glued together while preserving the constant K:

Theorem 4. Let U be a region in $\mathbb{C}$ and $\phi : U \to V$ an orientation preserving homeomorphism. Assume that C is a closed contour in C such that C divides U into components U_1, U_2. If $\phi|_{U_1}$ and $\phi|_{U_2}$ are K-quasiconformal then so is ϕ.

2.3 Criterion for quasiconformality in the continuously differentiable case

In the case of continuously differentiable functions, there is an easier way to check for quasiconformality. The following criteria is useful in the proof of the circle packing theorem. It shows that K-quasiconformality is equivalent to a condition of the directional derivative $D_v f$ which is defined for $v \in S^1$ as

$$D_v\phi(z_0) := \frac{\partial}{\partial t}\phi(z_0 + tv)|_{t=0}.$$

Theorem 5. [4] Let $K \geq 1$ and $\phi : U \to V$ an orientation-preserving diffeomorphism. Then TFAE

(i) ϕ is K-quasiconformal.

(ii) For $z_0 \in U$, $v, w \in S^1$,

$$|D_v\phi(z_0)| \leq K|D_w\phi(z_0)|.$$

3 Circle Packings

Definition 3.1. Let R be a subset of the plane or the 2-sphere. A *circle packing* of R is a collection of closed circles in R with disjoint interiors. The union of the circles is assumed to be connected.

Therefore, in the plane, if we let $C_j = \{z \in \mathbb{C} : |z - z_j| = r_j\}$ with $C_j^\circ \cap C_k^\circ = \emptyset$, a circle packing is the collection $(C_j)_{j \in J}$.

The *nerve* of a circle packing is the embedded graph whose vertex set consists of the centers of the circles with edges joining the centers of tangent circles and passing through the point of tangency.

The circle packing theorem states that every connected planar graph can be realized as the nerve of a circle packing. In fact, given one additional assumption, this circle packing is unique:

Theorem 6 (**Andreev, Thurston [4]**)**.** If a planar graph is maximal (no edges can be added while preserving planarity) then it is the nerve of a circle packing which is unique up to reflections and Möbius transformations.

3.1 Hexagonal circle packing

An important example of a circle packing is called *regular hexagonal.* First, we need the hexagonal lattice

$$\Gamma := \{2n + 2e^{2\pi i/3}m : n, m \in \mathbb{Z}\}.$$

If $z_0 + S^1$ denotes the unit circle centered at z_0, the hexagonal circle packing is

$$\mathcal{H} := \{z_0 + S^1\}_{z_0 \in \Gamma}.$$

4 Approximation Conjecture

It was a conjecture of Thurston that one could approximate the conformal equivalence of the Riemann mapping theorem using circle packings. The conjecture was proven in [1] using quasiconformal maps. In this section we explain how this approximation works. In Section 5, we explain in more detail how to construct the approximation.

Let R be a simply connected region in the plane as in the statement of the Riemann mapping theorem. Use the following scheme to approximate the conformal equivalence ϕ guaranteed by the Riemann mapping theorem.

1. Use a regular hexagonal circle packing to fill R with circles.
2. Surround the circles with some Jordan curve.
3. By Theorem 6, this packing is a reflection/Möbius transform of a packing of the unit disk, with the Jordan curve corresponding to the unit circle.
4. The map which takes circles from the packing of R to circles in the packing of the unit disk is an approximation of the Riemann mapping.

Letting the radii of the circles go to 0 gives a better approximation of the Riemann map.

5 Circle packing maps converge

In order to give a more precise description of the above approximation scheme we need to first explain in more detail how to fill the region R with circles.

Let R be a simply connected bounded subset of $\mathbb{C}$ with distinguished points z_0 and z_1. Let $\mathcal{H}_\epsilon$ be the regular hexagonal packing of the plane by circles of radius ϵ. The *flower* of a circle in $\mathcal{H}_\epsilon$ is the region consisting of the circle, the six circles surrounding it, as well as the space in between the circles. Let C_0 be a circle in $\mathcal{H}_\epsilon$ whose flower contains z_0. Form all chains $C_0, C_1, \ldots, C_k$ of consecutively tangent circles in $\mathcal{H}_\epsilon$ emanating from C_0 and contained in R. Circles that appear in some such chain are called *inner circles* and the set of inner circles is denoted $\mathcal{I}_\epsilon$.

Circles tangent to some inner circle will be called *border circles.* The set $\mathcal{B}_\epsilon$ of border circles forms a cycle enclosing the inner circles called the border.

Let $C_\epsilon = \mathcal{I}_\epsilon \cup \mathcal{B}_\epsilon$ be the circle packing consisting of border and inner circles.

Let T_ϵ denote the planar graph which is the nerve of C_ϵ.

By Theorem 6, T_ϵ is also the nerve of a circle packing C'_ϵ of $\mathbb{D}$, the unit disk. There is thus a correspondence $C \to C'$ of circles C in C_ϵ with circles C' in C'_ϵ. Recall that we have two distinguished points $z_0, z_1 \in R$. Assume that C_1 is a circle with flower containing z_1. By applying a set of Möbius transformations we can assume that C'_0 is centered at the center of $\mathbb{D}$ and that C'_1 is centered on the real line.

The centers of tangent circles in C_ϵ form solid closed triangles which are equilateral since the circles came from a hexagonal packing. Let U_ϵ be the union of all these equilateral triangles. Similarly, let D_ϵ be the union of the solid closed triangles formed by centers of tangent circles in C'_ϵ. Note that triangles in D_ϵ are not in general equilateral. There is a piecewise affine map $\phi_\epsilon : U_\epsilon \to D_\epsilon$ mapping triangles to triangles. It is affine on each triangle.

We want to show that the maps ϕ_ϵ converge in some sense to the Riemann mapping ϕ. As a preliminary step, we note that the domains converge:

Lemma 5.1. *U_ϵ converges to U in the Hausdorff sense as $\epsilon \to 0$. Similarly, D_ϵ converges to $\mathbb{D}$ in the Hausdorff sense as $\epsilon \to 0$.*

We can now give a precise description of the approximation of the Riemann map. Let U be a simply connected region with distinguished points z_0, z_1. Recall once again that the Riemann mapping theorem guarantees the existence of a unique conformal map $\phi : U \to \mathbb{D}$ which maps $z_0 \mapsto 0$ and $z_1 \mapsto x$, x a positive real.

Theorem 7. *Let ϕ_ϵ be the piecewise affine map defined above. As $\epsilon \to 0$, ϕ_ϵ converges locally uniformly to ϕ.*

Before outlining the proof we need one more result:

Proposition 5.1. *[4] Let $\phi_n : U \to V_n$ be a sequence of K-quasiconformal maps for some $K \geq 1$, such that all the V_n are uniformly bounded. Then the ϕ_n are a normal family, that is, every sequence in ϕ_n contains a subsequence that converges locally uniformly.*

Proof. First we show that the ϕ_n restricted to compact subsets of U are equicontinuous: Let C be a compact subset of U and $z, w \in C$. The points z, w can be surrounded with an annulus $A = \{z : r < |z - z_0| < R\}$. The modulus of this annulus is given by $\log \frac{R}{r}$ (see Fact 1 in section 2.1). Therefore, letting $|z - w| \to 0$, the annulus surrounding z, w can have arbitrarily large modulus. Since modulus is preserved under conformal mapping, $\phi_n(A)$ has the same modulus. Furthermore, $\phi_n(A)$ surrounds $\phi_n(z)$ and $\phi_n(w)$. However, since $\phi_n(z), \phi_n(w) \in \phi_n(C)$, a bounded region, the distance $|\phi_n(z) - \phi_z(w)|$ can be forced to be arbitrarily small. Thus ϕ_n are equicontinuous on C. Equicontinuity on compact subsets gives locally uniform convergence of some subsequence.

Below we give only an outline the proof of Theorem 7. For a more complete proof see [4].

Proof outline for Theorem 7: Using a result called the Hexagonal Packing Lemma (see [4]) the maps ϕ_ϵ map equilateral triangles to triangles in D_ϵ which are arbitrarily close to equilateral. To be more precise, this means that the angles of the triangles of D_ϵ are $\frac{\pi}{3} + f(\epsilon)$ where $f(\epsilon) \to 0$ as $\epsilon \to 0$. Using this fact along with Theorem 5 one can then show that ϕ_ϵ is $1 + f(\epsilon)$-quasiconformal on each triangle of R. By our result on gluing quasiconformal maps (Theorem 4), this means it is actually $1 + f(\epsilon)$-quasiconformal on all of R.

Using Proposition 5.1, one can show that ϕ_ϵ converges locally uniformally to a limit function.

By Theorem 3 the limit function is conformal. The Riemann mapping theorem guarantees that there is a unique such function. Since we showed convergence on fixed compact subsets R of U we get that ϕ_ϵ converges locally uniformly to ϕ.

References

[1] Beardon, Alan F.; Stephenson, Kenneth The Schwarz-Pick lemma for circle packings. Illinois J. Math. 35 (1991), no. 4, 577606.

[2] Y. Imayoshi and M. Taniguchi. An Introduction to Teichmuller Spaces. Springer-Verlag. Tokyo (1992).

[3] Schramm, Oded, Rigidity of infinite (circle) packings. J. Amer. Math. Soc. 4 (1991), no. 1, 127149

[4] Tao, Terence, Circle packings, conformal maps, and quasiconformal maps. Blog Post, 12 April, 2018