Riemann-Roch Theorem

MAT 205B: Complex Analysis

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Introduction

Roughly speaking, the Riemann-Roch theorem gives us the number of linearly independent meromorphic functions on a compact connected Riemann surface under certain restrictions on the poles. It tells us that this number only depends on the genus of the surface, a easily computable number that depends on restrictions placed on the configuration of the poles and a correction term called the dimension of the 1st cohomology of the surface. The proof presented here uses the algebraic machinery of sheaves and cohomology of sheaves. We explain these notions succinctly in sections 1,2,3 and prove the main theorem in section 4. Finally, in section 5 we give an application. Most of the proofs presented here are taken from Forster, Otto. *Lectures on Riemann Surfaces.* Springer, 1981.

1 Sheaf of Abelian Groups

Definition 1.1 (presheaf)**.** Suppose X is a topological space and $\mathfrak{T}$ is the collection of open sets in X. A *presheaf* of abelian groups on X is a pair $(\mathscr{F}, \rho)$ consisting of

(i) a family $\mathscr{F} = (\mathscr{F}(U))_{U\in\mathfrak{T}}$ of abelian groups

(ii) a family $\rho = (\rho_V^U)_{U,V\in\mathfrak{T}}$ of group homomorphisms with the following properties

  (a) $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ where V is open in U

  (b) $\rho_U^U = id_{\mathscr{F}(U)}$ for every $U \in \mathfrak{T}$

  (c) $\rho_W^V \circ \rho_V^U = \rho_W^U$ for $W \subset V \subset U$

Generally one writes $\mathscr{F}$ instead of $(\mathscr{F}, \rho)$. The homomorphisms ρ_V^U are called restriction homomorphisms. In our applications, these homomorphisms will be actual restrictions. So, from now on instead of writing $\rho_V^U(f)$ for $f \in \mathscr{F}(U)$, we will write $f|V$

Definition 1.2 (sheaf)**.** A presheaf $\mathscr{F}$ on a topological space X is called a *sheaf* if for every open set $U \in X$ and every family of open subsets $U_i \subset U, i \in I$ that cover U $(U = \cup_{i\in I} U_i)$, the following axioms are satisfied:

(i) If $f, g \in \mathscr{F}(U)$ such that $f|U_i = g|U_i$ for all $i \in I$, then $f = g$.

(ii) Given $f_i \in \mathscr{F}(U_i)$ such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all $i, j \in I$, there exists an $f \in \mathscr{F}(U)$ such that $f_i = f|U_i$ for all $i \in I$

Example 1.3.

(a) Let X be any topological space. For an open set $U \subset X$ define $\mathscr{C}(U)$ to be the set of continuous functions $f : U \to \mathbb{C}$. $\mathscr{C}(U)$ is an abelian group under pointwise addition. For an open set $V \subset U$ define $\rho_V^U : \mathscr{C}(U) \to \mathscr{C}(V)$ by the usual restriction $\rho_V^U(f) = f|_V$. Then, clearly $(\mathscr{C}, \rho)$ is a presheaf. Moreover, it also satisfies the two sheaf axioms. If

$U_i \subset U$, $U = \cup_{i\in I} U_i$ and $f|U_i = g|U_i$ for all i then $f = g$ as $f|U_i, g|U_i$ are the usual restrictions. Next, if $f_i \in \mathscr{C}(U_i)$ and $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ then define $f : U \to \mathbb{C}$ by $f(x) = f_i(x)$ if $x \in U_i$. This map is well defined. Moreover for any $x \in U$, if $x \in U_i$ then $f|U_i = f_i$ is continuous. This means f is continuous at x. Therefore, $f \in \mathscr{C}(U)$ and the second axiom is also satisfied. Therefore, $\mathscr{C}$ is a sheaf.

(b) Suppose X is a Riemann surface. For any open set $U \in X$, define $\mathcal{O}(U)$ to be the group of holomorphic functions $f : U \to \mathbb{C}$ under addition. For open set $V \subset U$, again define $\rho_V^U(f) = f|_V$ to be the usual restriction. Then, for reasons similar to above, $\mathcal{O}$ is a sheaf

Definition 1.4. Suppose $\mathscr{F}$ is a presheaf on a topological space X. Then, for $a \in X$, we define the equivalence relation $\underset{a}{\sim}$ on $\coprod_{U\ni a} \mathscr{F}(U)$ as follows. If $f \in \mathscr{F}(U), g \in \mathscr{F}(V)$ then $f \underset{a}{\sim} g$ if there exists an open set W such that $a \in W \subset V \cap U$ and $f|W = g|W$

Definition 1.5 (stalk at a point and germs)**.** Let $\mathscr{F}$ be a presheaf on a topological space X. Then, for $a \in X$, the *stalk* of $\mathscr{F}$ at a is given by

$$\mathscr{F}_a = \coprod_{U\ni a} \mathscr{F}(U) \Big/ \underset{a}{\sim}$$

Let $\rho_a : \coprod_{U\ni a} \mathscr{F}(U) \to \coprod_{U\ni a} \mathscr{F}(U)/\underset{a}{\sim}$ be the quotient map. Then, for $f \in \mathscr{F}(U)$, $\rho_a(f)$ is called the germ of f at a. One can define addition on the stalk at a point in the following way. If $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(U')$ then define $\rho_a(f) + \rho_a(g) = \rho_a(f|U \cap U' + g|U \cap U')$. This makes $\mathscr{F}_a$ an abelian group.

2 Cohomology groups

Definition 2.1 (cochain group)**.** Suppose X is topological space and $\mathscr{F}$ is a sheaf of abelian groups on X. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover for X. Then for $q = 0, 1, 2, \ldots$, we define the q^{th} *cochain group* of $\mathscr{F}$ with respect to $\mathfrak{U}$ as

$$C^q(\mathfrak{U}, \mathscr{F}) = \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

Definition 2.2 (coboundary map)**.** Let $\mathfrak{U}, \mathscr{F}, X$ be as above. We then define the boundary operators

$$\delta : C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$$

$$\delta : C^1(\mathfrak{U}, \mathscr{F}) \to C^2(\mathfrak{U}, \mathscr{F})$$

as follows:

(i) For $(f_i)_{i\in I} \in C^0(\mathfrak{U}, \mathscr{F})$, let $\delta((f_i)_{i\in I}) = (g_{ij})_{i,j\in I}$ where

$$g_{ij} = f_j|U_i \cap U_j - f_i|U_i \cap U_j \in \mathscr{F}(U_i \cap U_j)$$

For simplicity of notation, we will write $g_{ij} = f_j - f_i$ from now on

(ii) For $(f_{ij})_{i,j\in I} \in C^1(\mathfrak{U}, \mathscr{F})$, let $\delta((f_{ij})_{i,j\in I}) = (g_{ijk})_{i,j,k\in I}$ where

$$g_{ijk} = f_{jk}|U_i \cap U_j \cap U_k - f_{ik}|U_i \cap U_j \cap U_k + f_{ij}|U_i \cap U_j \cap U_k$$

Again, for simplicity, we will write $g_{ijk} = f_{jk} - f_{ik} + f_{ij}$

These coboundary operators are homomorphisms since the restriction homomorphisms are homomorphisms.

Definition 2.3 (Cohomology w.r.t. a covering)**.** Let

$$Z^1(\mathfrak{U}, \mathscr{F}) := Ker(C^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^2(\mathfrak{U}, \mathscr{F}))$$

$$B^1(\mathfrak{U}, \mathscr{F}) := Im(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F}))$$

The elements of $Z^1(\mathfrak{U}, \mathscr{F})$ are called *1-cocycles* and the elements of $B^1(\mathfrak{U}, \mathscr{F})$ are called *1-coboundaries*. Coboundaries are also called splitting cocycles. One can easily check that every 1-coboundary is a 1-cocyle. The quotient map

$$H^1(\mathfrak{U}, \mathscr{F}) := Z^1(\mathfrak{U}, \mathscr{F})/B^1(\mathfrak{U}, \mathscr{F})$$

is called the *1st Cohomology group* with coefficient in $\mathscr{F}$ with respect to the covering $\mathfrak{U}$. Its elements are called Cohomology classes and two cocyles which belong to the same cohomology class are called *cohomologous*. Thus two cocycles are cohomologous precisely if their difference is a coboundary. The cohomology group defined above depends on the covering. In order to have cohomology groups which depend only on X and $\mathscr{F}$, one has to use finer and finer converings and then take a “limit”. We will make this idea precise now.

Lemma 2.4. If (f_{ij}) is a cocycle then $f_{ii} = 0$ and $f_{ij} = -f_{ji}$

Proof. Since (f_{ij}) is a cocycle, we have

$$f_{ik} = f_{ij} + f_{jk} \quad \textit{on } U_i \cap U_j \cap U_k$$

Setting $i = j = k$, we get $f_{ii} = 0$. Then, setting $i = k$ gives us $f_{ij} = -f_{ji}$ □

Definition 2.5. An open covering $\mathfrak{B} = (V_k)_{k\in K}$ is called finer than the covering $\mathfrak{U} = (U_i)_{i\in I}$ (denoted $\mathfrak{B} < \mathfrak{U}$) if there exists a map $\tau : K \to I$ such that $V_k \subset U_{\tau k}$ for every $k \in K$. This just means that every V_k is contained in some U_i. Now, define the homomorphism

$$t^{\mathfrak{U}}_{\mathfrak{B}} : Z^1(\mathfrak{U}, \mathscr{F}) \to Z^1(\mathfrak{B}, \mathscr{F})$$

in the following way. For $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ let $t^{\mathfrak{U}}_{\mathfrak{B}}((f_{ij})) = (g_{kl})$ where

$$g_{kl} := f_{\tau k, \tau l} | V_k \cap V_l$$

It can be easily checked that this map takes coboundaries to coboundaries. Thus it induces a homomorphism of the cohomology groups $H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{B}, \mathscr{F})$, which we also denote by $t^{\mathfrak{U}}_{\mathfrak{B}}$

Lemma 2.6. The mapping $t^{\mathfrak{U}}_{\mathfrak{B}}$ is independent of the choice of $\tau : K \to I$

Proof. Suppose $\tilde{\tau} : K \to I$ is another mapping such that $V_k \subset U_{\tilde{\tau} k}$ for every $k \in K$. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ and let

$$g_{kl} := f_{\tau k, \tau l} | V_k \cap V_l \quad \text{and} \quad \tilde{g}_{kl} := f_{\tilde{\tau} k, \tilde{\tau} l} | V_k \cap V_l$$

We need to show that the cocycles (g_{kl}) and $(\tilde{g}_{kl})$ are cohomologous. Since $V_k \subset U_{\tau k} \cap U_{\tilde{\tau} k}$, one can define

$$h_k := f_{\tau k, \tilde{\tau} k} | V_k \in \mathscr{F}(V_k)$$

Then, on $V_k \cap V_l$ we have

$$\begin{aligned} g_{kl} - \tilde{g}_{kl} &= f_{\tau k, \tau l} - f_{\tilde{\tau} k, \tilde{\tau} l} \\ &= f_{\tau k, \tau l} + f_{\tau l, \tilde{\tau} k} - f_{\tau l, \tilde{\tau} k} - f_{\tilde{\tau} k, \tilde{\tau} l} \\ &= f_{\tau k, \tilde{\tau} k} - f_{\tau l, \tilde{\tau} l} \\ &= h_k - h_l \end{aligned}$$

Thus, $(g_{kl}) - (\tilde{g}_{kl})$ is a coboundary. □

Lemma 2.7. The mapping

$$t^{\mathfrak{U}}_{\mathfrak{B}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{B}, \mathscr{F})$$

is injective

Proof. Suppose $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$ is a cocycle whose image is a coboundary. We need to show that (f_{ij}) itself is a coboundary. Say $f_{\tau k, \tau l} = g_k - g_l$ on $V_k \cap V_l$ where $g_k \in \mathscr{F}(V_k)$. Then on $U_i \cap V_k \cap V_l$ one has

$$g_k - g_l = f_{\tau k, \tau l} = f_{\tau k, i} + f_{i, \tau l} = f_{i, \tau l} - f_{i, \tau k}$$

and thus $f_{i, \tau k} + g_k = f_{i, \tau l} + g_l$. Applying definition 1.2(ii) we get $h_i \in \mathscr{F}(U_i)$ such that

$$h_i = f_{i, \tau k} + g_k \quad on\ U_i \cap V_k$$

Therefore, we have

$$f_{ij} = f_{i, \tau k} + f_{\tau k, j} = f_{i, \tau k} + g_k - f_{j, \tau k} - g_k = h_i - h_j$$

Since k is arbitrary, it follows from definition 1.2(i) that this equation is valid over $U_i \cap U_j$. Therefore, (f_{ij}) is a coboundary. □

Remark. If one has open coverings $\mathfrak{W} < \mathfrak{B} < \mathfrak{U}$ then from 2.6 it follows that

$$t^{\mathfrak{B}}_{\mathfrak{W}} \circ t^{\mathfrak{U}}_{\mathfrak{B}} = t^{\mathfrak{U}}_{\mathfrak{W}}$$

Definition 2.8. Define an equivalence relation $\sim$ on the disjoint union of the $H^1(\mathfrak{U}, \mathscr{F})$, where $\mathfrak{U}$ runs through all open coverings of X in the following way. Two cohomology classes $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ and $\eta \in H^1(\mathfrak{U}', \mathscr{F})$ are equivalent if there exists an open covering $\mathfrak{B}$ with $\mathfrak{B} < \mathfrak{U}$ and $\mathfrak{B} < \mathfrak{U}'$ such that $t^{\mathfrak{U}}_{\mathfrak{B}}(\xi) = t^{\mathfrak{U}'}_{\mathfrak{B}}(\eta)$. The set of equivalence classes

$$H^1(X, \mathscr{F}) := \left(\coprod_{\mathfrak{U}} H^1(\mathfrak{U}, \mathscr{F}) \right) \Big/ \sim$$

is called the *1st Cohomology group* of X with coefficients in $\mathscr{F}$. Addition in $H^1(X, \mathscr{F})$ is defined in the following way. For $x, y \in H^1(X, \mathscr{F})$ represented by $\xi \in H^1(\mathfrak{U}, \mathscr{F})$ and $\eta \in H^1(\mathfrak{U}', \mathscr{F})$, let $\mathfrak{B}$ be a common refinement of $\mathfrak{U}$ and $\mathfrak{U}'$. Then $x + y$ is defined to be the equivalence class of $t^{\mathfrak{U}}_{\mathfrak{B}}(\xi) + t^{\mathfrak{U}'}_{\mathfrak{B}}(\eta) \in H^1(\mathfrak{B}, \mathscr{F})$. One can check that this defnition is independent of the various choices made and makes $H^1(X, \mathscr{F})$ into an abelian group.

Remark. We've dealt with sheaves of abelian groups till now. However, one can see that all that we've been doing can be extended to sheaves of vector spaces as well. In this case $H^1(X, \mathscr{F})$ will be a vector space instead.

Definition 2.9 (Zeroth Cohomology Group)**.** Suppose $\mathscr{F}$ is a sheaf of abelian groups on the topological space X and $\mathfrak{U} = (U_i)_{i \in I}$ is an open covering of X. Let

$$\begin{aligned} Z^0(\mathfrak{U}, \mathscr{F}) &:= Ker(C^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathfrak{U}, \mathscr{F})), \\ B^0(\mathfrak{U}, \mathscr{F}) &:= \{0\}, \\ H^0(\mathfrak{U}, \mathscr{F}) &:= Z^0(\mathfrak{U}, \mathscr{F}) / B^0(\mathfrak{U}, \mathscr{F}) = Z^0(\mathfrak{U}, \mathscr{F}) \end{aligned}$$

$(f_i) \in Z^0(\mathfrak{U}, \mathscr{F})$ precisely if $f_i | U_i \cap U_j = f_j | U_i \cap U_j$ for every $i, j \in I$. By the sheaf axiom 1.2(ii) we can the conclude that there is a $f \in \mathscr{F}(X)$ s.t. $f|U_i = f_i$ for every $i \in I$. Therefore there is a natural isomorphism

$$H^0(\mathfrak{U}, \mathscr{F}) = Z^0(\mathfrak{U}, \mathscr{F}) \cong \mathscr{F}(X)$$

Thus, the groups $H^0(\mathfrak{U}, \mathscr{F})$ are independent of $\mathfrak{U}$ and one can define

$$H^0(X, \mathscr{F}) := \mathscr{F}(X)$$

3 The exact Cohomology Sequence

Definition 3.1. Suppose $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups on the topological space X. A *sheaf homomorphism* $\alpha : \mathscr{F} \to \mathscr{G}$ is a family of group homomorphisms

$$\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U), \quad U \text{ open in } X$$

such that for every pair of open sets $U, V \subset X$ with $V \subset U$, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\alpha_U} & \mathscr{G}(U) \\ {\scriptstyle restr.}\downarrow & & \downarrow{\scriptstyle restr.} \\ \mathscr{F}(V) & \xrightarrow{\alpha_V} & \mathscr{G}(V) \end{array}$$

restr. denotes the restriction homomorphisms. If all the α_U are isomorphisms then α is called and isomorphism. One often writes $\alpha : \mathscr{F}(U) \to \mathscr{G}(U)$ instead of $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$. Also, note that analogous definitions exist for sheaves of vector spaces.

Lemma 3.2. Suppose $\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf homomorphism on X. For $f \in \mathscr{F}(U)$ and $g \in \mathscr{G}(V)$ let $\rho_x(f)$ and $\eta_x(g)$ be the corresponding germs at x. Then, the map $\alpha_x : \mathscr{F}_x \to \mathscr{G}_x$ given by

$$\alpha_x(\rho_x(f)) := \eta_x(\alpha_U(f))$$

is a homomorphism.

Proof. To check that the map is well defined, let $f' \in \mathscr{F}(U')$ such that $\rho_x(f') = \rho_x(f)$. We need to check that $\eta_x(\alpha_U(f)) = \eta_x(\alpha_{U'}(f'))$. i.e. we need to show that there is an open set $W \subset U \cap U'$ such that $\alpha_U(f)|W = \alpha_{U'}(f')|W$. Since $\rho_x(f) = \rho_x(f')$, we get a $W \subset U \cap U'$ such that $f|W = f'|W$. Then, $\alpha_U(f)|W = \alpha_W(f|W) = \alpha_W(f'|W) = \alpha_{U'}(f')|W$.
Next, note that for $f' \in \mathscr{F}(U')$, $\alpha_x(\rho_x(f) + \rho_x(f')) = \alpha_x(\rho_x(f|U \cap U') + \rho_x(f'|U \cap U') = \eta_x(\alpha_{U\cap U'}(f|U\cap U' + f'|U\cap U')) = \eta_x(\alpha_{U\cap U'}(f|U\cap U')) + \eta_x(\alpha_{U\cap U'}(f'|U\cap U')) = \alpha_x(\rho_x(f)) + \alpha_x(\rho_x(f'))$. And, $\alpha_x(-\rho_x(f)) = \alpha_x(\rho_x(-f)) = \eta_x(\alpha_U(-f)) = -\eta_x(\alpha_U(f))$ □

Definition 3.3. A sequence of sheaf homomorphism $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is called exact if for each $x \in X$ the sequence

$$\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$$

is exact. $\alpha : \mathscr{F} \to \mathscr{G}$ is called a *monomorphism* if $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G}$ is exact and an *epimorphism* if $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \to 0$ is exact.

Lemma 3.4. Suppose $\alpha : \mathscr{F} \to \mathscr{G}$ is a sheaf monomorphism on the topological space X. Then for every subset $U \subset X$ the mapping $\alpha_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

Proof. Suppose $f \in \mathscr{F}(U)$ and $\alpha_U(f) = 0$. Then, for every $x \in U$, $\alpha_x(\rho_x(f)) = 0$. Therefore, by injectivity of α_x, we have $\rho_x(f) = 0$ for every $x \in U$. This means that for every $x \in U$, there exists a neighborhood $V_x \subset U$ such that $f|V_x = 0$. But then by sheaf axiom 1.2(i) it follows that $f = 0$ □

Lemma 3.5. Suppose $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ is an exact sequence of sheaves on X. Then for every open set $U \subset X$ the sequence

$$0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U) \xrightarrow{\beta} \mathscr{H}(U)$$

is exact

Proof.

(a) The exactness of $0 \to \mathscr{F}(U) \xrightarrow{\alpha} \mathscr{G}(U)$ was shown in 3.4

(b) To show $Im(\alpha) \subset Ker(\beta)$ suppose $f \in \mathscr{F}(U)$ and $g = \alpha(f)$. Since $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$ is exact, we can conclude that the germ of g at x is in the kernel of β_x for every $x \in U$. This just means that for every $x \in X$, there exists an open set $V_x \subset U$ such that $\beta(g)|V_x = 0$. Then, by the sheaf axiom 1.2(i), $\beta(g) = 0$

(c) To prove inclusion $Ker\ \beta \subset Im\ \alpha$ suppose $g \in \mathscr{G}(U)$ with $\beta(g) = 0$. Since for every $x \in U$ one has $Ker\ \beta_x = Im\ \alpha_x$, there is an open covering $(V_i)_{i \in I}$ of U and elements $f_i \in \mathscr{F}(V_i)$ such that $\alpha(f_i) = g|V_i$ for every $i \in I$. On the the intersection $V_i \cap V_j$ one has $\alpha(f_i - f_j) = g|V_i \cap V_j - g|V_i \cap V_j = 0$. Hence by lemma 3.4 it follows that $f_i = f_j$ on $V_i \cap V_j$. Then, by sheaf axiom 1.2(ii) there exists $f \in \mathscr{F}(U)$ with $f|V_i = f_i$ for every $i \in I$. Since $\alpha(f)|V_i = \alpha(f|V_i) = g|V_i$, it follows from the sheaf axiom 1.2(i) applied to $\mathscr{G}$ that $\alpha(f) = g$.

□

Definition 3.6. Any homomorphism $\alpha : \mathscr{F} \to \mathscr{G}$ of sheaves on the topological space X induces homomorphisms

$$\alpha^0 : H^0(X, \mathscr{F}) \to H^0(X, \mathscr{G}),$$

$$\alpha^1 : H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G})$$

The homomorphism α^0 is nothing but $\alpha_X : \mathscr{F}(X) \to \mathscr{G}(X)$. The homomorphism α^1 can be constructed as follows. Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. Consider the mapping

$$\alpha_{\mathfrak{U}} : C^1(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{G})$$

which assigns to each co chain $\xi = (f_{ij}) \in C^1(\mathfrak{U}, \mathscr{F})$ the cochain

$$\alpha_{\mathfrak{U}}(\xi) := (\alpha(f_{ij})) \in C^1(\mathfrak{U}, \mathscr{G})$$

One can check that this mapping takes cocycles to cocycles and coboundaries to coboundaries and thus induces a homomorphism $\tilde{\alpha}_{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{F}) \to H^1(\mathfrak{U}, \mathscr{G})$. The collection of $\tilde{\alpha}_{\mathfrak{U}}$, where $\mathfrak{U}$ runs over all open coverings of X, then induces the homomorphism α^1.

Definition 3.7 (The Connecting Homomorphism)**.** Suppose $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ is an exact sequence of sheaves on the topological space X. The connecting homomorphism

$$\delta^* : H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F})$$

is defined as follows. Suppose $h \in H^0(X, \mathscr{H}) = \mathscr{H}(X)$. Since all homomorphisms $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ are surjective, there exists an open covering $\mathfrak{U} = (U_i)_{i \in I}$ of X and a cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{G})$ such that

$$\beta(g_i) = h|U_i \quad \text{for every } i \in I$$

Hence, $\beta(g_j - g_i) = 0$ on $U_i \cap U_j$. By lemma 3.5 there exits $f_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that

$$\alpha(f_{ij}) = g_j - g_i$$

On $U_i \cap U_j \cap U_k$ one has $\alpha(f_{ij} + f_{jk} - f_{ik}) = 0$. Thus by lemma 3.4 $f_{ij} + f_{jk} - f_{ik} = 0$, i.e.,

$$(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$$

Let $\delta^*(h)$ be the cohomology class represented by (f_{ij}). One can check that this definition is independent of the various choices made.

Theorem 3.8. Suppose X is a topological space and $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \to 0$ is a short exact sequence of sheaves on X. Then the induced sequence of cohomology groups

$$0 \to H^0(X, \mathscr{F}) \xrightarrow{\alpha^0} H^0(X, \mathscr{G}) \xrightarrow{\beta^0} H^0(X, \mathscr{H}) \xrightarrow{\delta^*}$$
$$\xrightarrow{\delta^*} H^1(X, \mathscr{F}) \xrightarrow{\alpha^1} H^1(X, \mathscr{G}) \xrightarrow{\beta^1} H^1(X, \mathscr{H})$$

is exact.

Proof.

(a) The exactness at $H^0(X, \mathscr{F})$ and $H^0(X, \mathscr{G})$ follows from lemma 3.5.

(b) To show $Im\ \beta^0 \subset Ker\ \delta^*$, let $g \in H^0(X, \mathscr{G})$ and $h = \beta^0(g)$. In the construction of $\delta^* h$ described in definition 3.7 one can choose $g_i = g|U_i$. But then $f_{ij} = 0$ and thus $\delta^* h = 0$.

(c) To show $Ker\ \delta^* \subset Im\ \beta^0$, suppose $h \in Ker\ \delta^*$. Using notation of definition 3.7 one can represent $\delta^* h$ by the cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Since $\delta^* h = 0$ there exists a cochain $(f_i) \in C^0(\mathfrak{U}, \mathscr{F})$ such that $f_{ij} = f_j - f_i$ on $U_i \cap U_j$. Set $\tilde{g}_i = g_i - \alpha(f_i)$. Then $\tilde{g}_i = \tilde{g}_j$ on $U_i \cap U_j$ since $\alpha(f_{ij}) = g_j - g_i$. Thus $\tilde{g}_i$ are restrictions of some global element $g \in H^0(X, \mathscr{G})$. On U_i one then has $\beta(g) = \beta(\tilde{g}_i) = \beta(g_i - \alpha(f_i)) = \beta(g_i) = h$, i.e. $h \in Im\ \beta^0$.

(d) $Im\ \delta^* \subset Ker\ \alpha^1$ follows from the fact that in definition 3.7, $\alpha(f_{ij}) = g_j - g_i$.

(e) To show $Ker\ \alpha^1 \subset Im\ \delta^*$ suppose $\xi \in Ker\ \alpha^1$ is represented by the cocycle $(f_{ij}) \in Z^1(\mathfrak{U}, \mathscr{F})$. Since $\alpha^1(\xi) = 0$, there exists a cochain $(g_i) \in C^0(\mathfrak{U}, \mathscr{G})$ such that $\alpha(f_{ij}) = g_j - g_i$ on $U_i \cap U_j$. This implies

$$0 = \beta(\alpha(f_{ij})) = \beta(g_j) - \beta(g_i) \quad on\ U_i \cap U_j$$

Therefore, there exists $h \in \mathscr{H}(X) = H^0(X, \mathscr{H})$ such that $h|U_i = \beta(g_i)$. The construction given in definition 3.7 shows that $\delta^* h = \xi$.

(f) $Im\ \alpha^1 \subset Ker\beta^1$ follows from the fact that

$$\mathscr{F}(U_i \cap U_j) \xrightarrow{\alpha} \mathscr{G}(U_i \cap U_j) \xrightarrow{\beta} \mathscr{H}(U_i \cap U_j)$$

is exact by lemma 3.5.

(g) To show $Ker\ \beta^1 \subset Im\ \alpha^1$ suppose $\eta \in Ker\ \beta^1$ is represented by the cocycle $(g_{ij}) \in Z^1(\mathfrak{U}, \mathscr{G})$, where $\mathfrak{U} = (U_i)_{i \in I}$. Then there is a cochain $(h_i) \in C^0(\mathfrak{U}, \mathscr{H})$ such that $\beta(g_{ij}) = h_j - h_i$. For every $x \in X$ choose $\tau x \in I$ such that $x \in U_{\tau x}$. Since $\beta_x : \mathscr{G}_x \to \mathscr{H}_x$ is surjective, there is an open neighborhood $V_x \subset U_{\tau x}$ of x and an element $g_x \in \mathscr{G}(V_x)$ such that $\beta(g_x) = h_{\tau x}|V_x$. Let $\mathfrak{B} = (V_x)_{x \in X}$ and $\tilde{g}_{xy} = g_{\tau x, \tau y}|V_x \cap V_y$. Then $(\tilde{g}_{xy}) \in Z^1(\mathfrak{B}, \mathscr{G})$ is a cocycle which also represents the cohomology class η. Let $\psi_{xy} := \tilde{g}_{xy} - g_y + g_x$. The cocycle (ψ_{xy}) is cohomologous to $(\tilde{g}_{xy})$ and $\beta(\psi_{xy}) = 0$. Thus there exists $f_{xy} \in \mathscr{F}(V_x \cap V_y)$ such that $\alpha(f_{xy}) = \psi_{xy}$. Since

$$\alpha : \mathscr{F}(V_x \cap V_y \cap V_z) \to \mathscr{G}(V_x \cap V_y \cap V_z)$$

is injective by lemma 3.4, $(f_{xy}) \in Z^1(\mathfrak{B}, \mathscr{F})$. Therefore the cohomology class $\xi \in H^1(X, \mathscr{F})$ of (f_{xy}) satisfies $\alpha^1(\xi) = \eta$. This completes the proof.

□

4 The Riemann-Roch Theorem

Definition 4.1. For a Riemann Surface X and an open set $U \subset X$ define $\mathcal{O}(U) = \{f : U \to \mathbb{C} : \text{f is holomorphic}\}$ and $\mathscr{M}(U) = \{f : U \to \mathbb{P}^1 : \text{f is holomorphic}\}$, where $\mathbb{P}^1$ is the riemann sphere. Then with the restriction homomorphism being the usual restriction, $\mathscr{M}, \mathcal{O}$ are sheafs on X. $\mathscr{M}$ is called the sheaf of meromorphic functions and $\mathcal{O}$ is called the sheaf of holomorphic functions on X.

Definition 4.2 (Divisors)**.** Let X be a Riemann surface. A divisor on X is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$. With respect to addition the set of all divisors on X is an abelian group which we denote $Div(X)$. For $D, D' \in Div(X)$ we say $D \leq D'$ if $D(x) \leq D'(x)$ for every $x \in X$

Definition 4.3 (Divisors of Meromorphic Functions)**.** Suppose X is a Riemann surface and U is an open subset of X. For meromorphic function $f \in \mathscr{M}(U)$ and $a \in U$ define

$$ord_a(f) := \begin{cases} 0, & \text{if f is holomorphic and non-zero at a,} \\ k, & \text{if f has a zero of order k at a,} \\ -k, & \text{if f has a pole of order k at a,} \\ \infty, & \text{if f is identically zero in a neighborhood of a.} \end{cases}$$

Thus for any meromorphic function $\mathscr{M}(X) \setminus \{0\}$, the mapping $x \mapsto ord_x(f)$ is a divisor on X. It is called the divisor of f and denoted (f).

Definition 4.4 (The Degree of a Divisor)**.** Suppose now that X is a compact Riemann surface. Then for every $D \in Div(X)$ there are only finitely many $x \in X$ such that $D(x) \neq 0$. Hence one can define

$$Deg\ D := \sum_{x \in X} D(x)$$

This mapping Deg is a group homomorphism

Definition 4.5. Suppose D is a divisor on a riemann surface X. For any open set $U \subset X$ define

$$\mathcal{O}_D(U) := \{f \in \mathscr{M}(U) : ord_x(f) \geq -D(x) \text{ for every } x \in X\}$$

Together with the usual restriction mappings, $\mathcal{O}_D$ is a sheaf. In the special case of $D = 0$, we get $\mathcal{O}_0 = \mathcal{O}$.

Lemma 4.6. If X is a connected compact Riemann surface and $f \in \mathcal{O}(X)$ then f is a constant map.

Proof. We will show that if f is non-constant then it must be open. Then, since X is compact, we can conclude that $f(X) \subset \mathbb{C}$ is both open and compact. No non-empty subset of $\mathbb{C}$ is both open and compact. So we reach a contradiction.
To show a non-constant $f \in \mathcal{O}(X)$ is open, let $\{\phi_i : U_i \to V_i \subset \mathbb{C}\}_{i \in I}$ be a an atlas on X. It is enough to show that $f|_{U_i}$ is open. If $f|_{U_i}$ is non-constant, then since ϕ_i^{-1} is a bijection and $f|_{U_i}$ is holomorphic, $f|_{U_i} \circ \phi_i^{-1}$ is a non-constant holomorphic function from $V_i \in \mathbb{C}$ to $\mathbb{C}$ and hence is open. Since ϕ_i^{-1} is a homeomorphism, this implies $f|_{U_i}$ is open.

So, assume that $f|_{U_i}$ is constant for some i. We will show that in this case f is a constant map. Define

$$S = \{x \in X : \text{there exists open set } U \subset X, x \in U \text{ such that } f|_U = 0\}$$

From the definition it is easy to see that S is open. To show it is closed, let (x_n) be a sequence in S that converges to some $x \in X$. If we take a chart $\phi : U \to V \subset \mathbb{C}$ around x then we can assume $x_n \in U$ as eventually all $x_n's$ will be in U. The map $f|_U \circ \phi^{-1}$ is zero at $\phi(x_n)$. i.e. we get a holomorphic map from V to $\mathbb{C}$ which has zeros that accumulate. This implies $f|_U \circ \phi^{-1}$ is identically zero $\implies f|_U$ is identically zero. Which in turn implies that $x \in S$. This proves S is both open and closed. If $f|_{U_i} = 0$ then S is non empty and thus $S = X$,i.e., $f = 0$.
To complete the proof note that if $f|_{U_i} = c$ then $(f - c)|_{U_i} = 0$ and hence $(f - c) = 0$, i.e., $f = c$. ∎

Definition 4.7 (Genus of a Riemann Surface)**.** Suppose X is a compact Riemann surface. Then

$$g := dim\ H^1(X, \mathcal{O})$$

is called the *genus* of X.

Definition 4.8 (The Skyscraper Sheaf $\mathbb{C}_p$)**.** Suppose p is a point on a Riemann surface X. Define a sheaf $\mathbb{C}_p$ on X by

$$\mathbb{C}_p(U) := \begin{cases} \mathbb{C} & \text{if } p \in U, \\ 0 & \text{if } p \notin U \end{cases}$$

Define the restriction maps in the following way. If $\mathbb{C}_p(U) = 0$ then define ρ_V^U to be the zero map. If $\mathbb{C}_p(V) = 0$ then again define ρ_V^U to be the zero map. If both $\mathbb{C}_p(U)$ and $\mathbb{C}_p(V)$ are $\mathbb{C}$ then define ρ_V^U to be the identity.

Lemma 4.9.

(i) $H^0(X, \mathbb{C}_p) \cong \mathbb{C}$

(ii) $H^1(X, \mathbb{C}_p) = 0$

Proof. $H^0(X, \mathbb{C}_p) \cong \mathbb{C}_p(X) = \mathbb{C}$. Therefore (i) is trivially true. To show (ii), consider a cohomology class $\xi \in H^1(X, \mathbb{C}_p)$ which is represented by a cocycle in $Z^1(\mathfrak{U}, \mathbb{C}_p)$. By taking intersection with $X \setminus \{p\}$ one can obtain a refinement $\mathfrak{B} = (V_\alpha)_{\alpha \in A}$ such that the point p is in only one V_α. Then, for $(f_{ij}) \in C^1(\mathfrak{B}, \mathbb{C}_p)$, if $i \neq j$ then $f_{ij} = 0$ since $p \notin V_i \cap V_j$. if $(f_{ij}) \in Z^1(\mathfrak{B}, \mathbb{C}_p)$ then $f_{ii} = 0$. Therefore, we conclude that $(f_{ij}) = 0$ and $Z^1(\mathfrak{B}, \mathbb{C}_p) = 0$ (hence $H^1(\mathfrak{B}, \mathbb{C}_p) = 0$). Since $t_{\mathfrak{B}}^{\mathfrak{U}}$ is injective, we conclude that $H^1(\mathfrak{U}, \mathbb{C}_p) = 0$. Since this is true for all open covers $\mathfrak{U}$, $H^1(X, \mathbb{C}_p) = 0$ □

Lemma 4.10. For $p \in X$, let P be the divisor that takes the value 1 at p and zero elsewhere. Then, we have the following exact sequence

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \\ \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0 \end{aligned}$$

Proof. It is easy to see that the natural inclusion $i : \mathcal{O}_D \to \mathcal{O}_{D+P}$ is an injection on the level of stalks. So $0 \to \mathcal{O}_D \to \mathcal{O}_{D+P}$ is exact. Let $z : V \to U \subset \mathbb{C}$ be a local coordinate on X around p such that $z(p) = 0$. Define a sheaf homomorphism

$$\beta : \mathcal{O}_{D+P} \to \mathbb{C}_p$$

as follows. If $p \notin U$, then $\mathbb{C}_p(U) = 0$. So set $\beta_U = 0$. In this case since $D = D + P$ on U, $\mathcal{O}_D(U) = \mathcal{O}_{D+P}(U)$. If $p \in U$ then $f \in \mathcal{O}_{D+P}(U)$ admits a laurent series around p in the coordinate z,

$$f = \sum_{n=-k-1}^{\infty} c_n z^n$$

where $k = D(p)$. Set

$$\beta_U(f) := c_{-k-1} \in \mathbb{C} = \mathbb{C}_p$$

On the level of stalks, we can check that $\beta_x = 0, (\mathbb{C}_p)_x = 0$ if $x \neq p$. if $x = p$, we have $\beta_p(\rho_p(f)) = c_{-k-1}$ and $(\mathbb{C}_p)_p = \mathbb{C}$. we can find a small coordinate neighborhood V around p where we can define $f \in \mathcal{O}_{P+D}$ via a power series such that $c_{-k-1} \neq 0$. For this f, $\beta_p(\rho_p(f)) \neq 0$ and thus β_p is surjective. Therefore, $\mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_p \to 0$ is exact.
Finally, for $f \in \mathcal{O}_{D+P}(U)$, where $p \in U$, $\beta_U(f) = 0$ if and only if $f \in Im(i)$. Therefore, $Ker(\beta_U) = Im(i)$. One can check that such a relation hold at the level of stalks as well. Thus, $\mathcal{O}_D \xrightarrow{i} \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_p$ is exact
Therefore, we conclude that

$$0 \to \mathcal{O}_D \xrightarrow{i} \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_p \to 0$$

is exact. Therefore, applying Theorem 3.8 and lemma 4.10, we get the following exact sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \\ &\to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0 \end{aligned}$$

□

Remark. All our analysis till now can be applied to sheaves of vector spaces as well. In particular, when applied to the case of vector spaces, the exact sequence obtained above is an exact sequence of vector spaces.

Theorem 4.11 (**The Riemann-Roch Theorem**)**.** Suppose D is a divisor on a compact connected Riemann Surface X of genus g. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and

$$dim\ H^0(X, \mathcal{O}_D) - dim\ H^1(X, \mathcal{O}_D) = 1 - g + deg\ D \tag{1}$$

Proof.

(a) If $D = 0$ then $H^0(X, \mathcal{O}_0) = \mathcal{O}$ consists of only constant functions by lemma 4.6 and thus $dim\ H^0(X, \mathcal{O}_0) = 1$. $dim\ H^1(X, \mathcal{O}_0) = g$ by definition. $Deg\ D = 0$. Therefore the above equation holds.

(b) Denote by P the divisor that takes value 1 at p and 0 elsewhere. Let $D' = D + P$. Also suppose that the result holds for one of the divisors D, D'. Define

$$\begin{aligned} V &:= Im(H^0(X, \mathcal{O}_{D'}) \to \mathbb{C}) \\ W &:= \mathbb{C}/V \end{aligned}$$

Then, $dim\ V + dim\ W = dim\ \mathbb{C} = 1 = deg\ D' - deg\ D$. Applying the analogue of lemma 4.10 for vector spaces, we see that

$$\begin{aligned} 0 &\to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D+P}) \to \mathbb{C} \\ &\to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D+P}) \to 0 \end{aligned}$$

is exact. Therefore, we have the following exact sequences.

$$0 \to H^0(X, \mathcal{O}_D) \to H^0(X, \mathcal{O}_{D'}) \to V \to 0$$
$$0 \to W \to H^1(X, \mathcal{O}_D) \to H^1(X, \mathcal{O}_{D'}) \to 0$$

From the exactness of these we get

$$H^0(X, \mathcal{O}_{D'}) \cong H^0(X, \mathcal{O}_D) \oplus V$$
$$H^1(X, \mathcal{O}_D) \cong H^1(X, \mathcal{O}_{D'}) \oplus W$$

Therefore if one of $H^0(X, \mathcal{O}_{D'}), H^0(X, \mathcal{O}_D)$ is finite dimensional, so is the other one. Similarly if one of $H^1(X, \mathcal{O}_D), H^1(X, \mathcal{O}_{D'})$ is finite dimensional, so is the other one. One can then write

$$dim\ H^0(X, \mathcal{O}_{D'}) = dim\ H^0(X, \mathcal{O}_D) + dim\ V$$
$$dim\ H^1(X, \mathcal{O}_D) = dim\ H^1(X, \mathcal{O}_{D'}) + dim\ W$$

Adding the two equations above and using $dim\ V + dim\ W = deg\ D' - deg\ D$, one gets

$$dim\ H^0(X, \mathcal{O}_{D'}) - dim\ H^1(X, \mathcal{O}_{D'}) - deg\ D' = dim\ H^0(X, \mathcal{O}_D) - dim\ H^1(X, \mathcal{O}_D) - deg\ D$$

Therefore, if equation (1) holds for one of D, D', then it hold for the other as well.

(c) An arbitrary divisor D can be written as

$$D = P_1 + P_2 + \cdots + P_m - P_{m+1} - P_{m+2} - \cdots - P_n$$

We proved the result for the base case $D = 0$ in (a). Thus, we can induct using (b) to prove the general case.

□

5 An application

Theorem 5.1. Suppose X is a compact connected Riemann surface of genus g and a is a point of X. Then there is a non-constant meromorphic function f on X which has a pole of order $\leq g + 1$ at a and is otherwise holomorphic.

Proof. Let D be the divisor that takes value $D(a) = g + 1$ and $D(x) = 0$ if $x \neq a$. Then, applying the Riemann-Roch theorem to this divisor, we get

$$dim\ \mathcal{O}_D(X) = dim\ H^0(X, \mathcal{O}_D) \geq 1 - g + deg\ D = 2$$

The constant functions in $\mathcal{O}_D$ form a one dimensional subspace. Therefore, there exists a non-constant function in $\mathcal{O}_D(X)$. This is the required function. □

We will use the above to show that every compact connected riemann surface of genus 0 is biholomorphic to the Riemann sphere $\mathbb{P}^1$. Before proving this, we need two additional facts which we will not prove.

Fact 1: Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic map such that the pre-image of every compact set is compact. Then there exists a natural number n such that f takes every value $c \in Y$, counting multiplicities, n times.
Fact 2: If X, Y are Riemann surfaces and $f : X \to Y$ is holomorphic and bijective. Then, f is a biholomorphism.

Corollary 5.1.1. Every Riemann-surface of genus 0 is biholomorphic to $\mathbb{P}^1$

Proof. Theorem 5.1 gives us a non-constant holomorphic map $f : X \to \mathbb{P}^1$ that assumes the value ∞ with multiplicity $\leq g + 1 = 1$. The multiplicity cannot be 0 since then we would have a holomorphic map on a compact connected Riemann surface, which by lemma 4.6 must be a constant. Next, observe that since X is compact, the pre-image of any compact set in $\mathbb{P}^1$ is compact. Therefore, by Fact 1, f assumes each value in $\mathbb{P}^1$ once. Therefore, f is bijective and by Fact 2, is a biholomophism. □

References

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