0.1 Introduction

We consider a simple polygon and ask whether there is a conformal map from $H$ to the polygon denoted by $P$. The answer is yes and such conformal map is written in terms of a ‘Schwarz-Christoffel’ integral. At the end, we will include Matlab Implementation and plots.

First, we define ‘Schwarz-Christoffel’ integral and build up to the theorem on defining a conformal map $F$ which maps $H$ conformally to $P$. We state the theorem here:

**Theorem 1.** If $F : H \to P$ is a conformal map from the upper half-plane to the polygonal region $P$ and maps the points $A_1, \ldots, A_{n-1}, \infty \in \mathbb{R}$ to the vertices of a polygon $p$, then there exists $C_1$ and $C_2$ such that

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2$$

(1)

where we will make these notation precise in the forthcoming discussion.

0.2 The Schwarz-Christoffel Integral

We define the **Schwarz-Christoffel integral** by

$$S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}}$$

(1)

where $A_1 < A_2 < \cdots < A_n$ and we assume the exponents satisfy $\beta_k < 1$ for all $k$ with $1 < \sum_{k=1}^{n} \beta_k$.

Now, we make sense of the integrand in (1). We define $(z - A_k)^{\beta_k}$ the branch (defined in the complex plane slit along the infinite ray $\{A_k + iy : y \leq 0\}$) which is positive when $z = x \in (A_k, \infty)$. Therefore,

$$(z - A_k)^{\beta_k} = \begin{cases} (x - A_k)^{\beta_k} & \text{if } x \text{ is real and } x > A_k, \\ |x - A_k|^{\beta_k} e^{i\pi \beta_k} & \text{if } x \text{ is real and } x < A_k, \end{cases}$$

(2)

By exercise 19 in [1], the complex plane slit along the union of the rays $\bigcup_{k=1}^{n} \{A_k + iy : y \leq 0\}$ (denote it by $\Omega$) is simply connected. Therefore, $S(z)$ is holomorphic on $\Omega$. Since
\( \beta_k < 1 \), this implies that we can integrate \((\zeta - A_k)^{-\beta_k}\) around \(A_k\) for \(k = 1, \ldots, n\). This means that \(S\) is continuous up to the real line, including \(A_k\). We have that \(S\) can be integrated along any path that avoids the open slits \(\bigcup_{k=1}^n \{A_k + iy : y < 0\}\).

For large \(|\zeta|\), there exists some positive constant \(c\) so that

\[
\left| \prod_{k=1}^n (\zeta - A_k)^{-\beta_k} \right| \leq c|\zeta|^{-\sum \beta_k} \tag{3}
\]

Given \(\sum_{k=1}^n \beta_k > 1\), this implies for sufficiently large \(|z|\), we have

\[
\int_{z}^{\infty} \frac{1}{|\zeta|^{\sum \beta_k}} d\zeta \tag{4}
\]

exists and is finite. To see this, integrating \(|\zeta|^{-\sum \beta_k}\) along a path starting at \(iy\) with \(y \gg 1\) to \(i\infty\) exists and is finite (by the \(p\)-test). Using this fact and Cauchy’s theorem imply that \(\lim_{r \to +\infty} S(re^{i\theta})\) exists and is independent of the angle \(\theta \in [0, 2\pi]\) (call this limit \(a_{\infty}\)). We let \(a_k := S(A_k)\) for \(k = 1, \ldots, n\).

We introduce a proposition that says that \(S\) maps \(\mathbb{R}\) onto the edges that bound a polygon whose vertices are given by \(a_1, \ldots, a_n\).

**Theorem 2.** Suppose \(S(z)\) is given by \((1)\).

(i) If \(\sum_{k=1}^n \beta_k = 2\), and \(p\) denotes the polygon whose vertices are given (in order) by \(a_1, \ldots, a_n\), then \(S\) maps the real axis onto \(p - \{a_{\infty}\}\) (meaning \(\mathbb{R}\) gets mapped into the edges of the polygon). The point \(a_{\infty}\) lies on the segment \([a_n, a_1]\) and is the image of the point at infinity. Also, the interior angle at the vertex \(a_k\) is \(\alpha_k \pi\) where \(\alpha_k = 1 - \beta_k\).

(ii) There is a similar conclusion when \(1 < \sum_{k=1}^n \beta_k < 2\), except now the image of the extended line \(\overline{\mathbb{R}}\) is the polygon of \(n + 1\) sides with vertices \(a_1, a_2, \ldots, a_n, a_{\infty}\). The angle at the vertex \(a_{\infty}\) is \(\alpha_{\infty} \pi\) where \(\alpha_{\infty} = 1 - \beta_{\infty}\), where \(\beta_{\infty} := 2 - \sum_{k=1}^n \beta_k\).
Figure 1: Action of the integral $S(z)$

Remember we are trying to find a conformal map from $\mathbb{H}$ to a given region $P$ that is bounded by a polygon, and the above theorem does not address this. We give two reasons for this.

- It is not true for general $n$ and generic choices of $A_1, \ldots, A_n$ that the image of $S$ under $R$ is a simple polygon. Also it is not generally true that the mapping $S$ is conformal on $\mathbb{H}$.

- The theorem above does not show that starting with a simply connected region $P$ whose boundary is a polygon $p$, the mapping $S$ is a conformal map from $\mathbb{H}$ to $P$ for certain choices of $A_1, \ldots, A_n$ and simple modifications. It turns out this is true and we will discuss this further.

0.3 Boundary Behavior

In this section, we consider a polygonal region $P$, a bounded simply connected open set whose boundary is a polygonal line $p$.

We like to study conformal maps from the half-plane $\mathbb{H}$ to $P$, so we consider studying conformal maps from the disk $D$ to $P$ (and their behaviors on $\partial D$). We introduce a theorem that discusses this.

**Theorem 3.** If $F : D \to P$ is a conformal map, then $F$ extends to a continuous bijection from the closure $\overline{D}$ of the disk $D$ to the closure $\overline{P}$ of the polygonal region. More specifically, $F$ is a bijection from $\partial D$ to the boundary polygon $p$.

The idea is to show that if $z_0 \in \partial D$, then $\lim_{z \to z_0} F(z)$ exists. We introduce three (3) lemmas (each depends on the previous lemma). See [1] for more detail on the proof. The first lemma we introduce, we assume $f : D \to \mathbb{C}$ is conformal.

**Lemma 4.** For $r \in (0, \frac{1}{2})$, denote $C_r$ by the circle centered at $z_0 \in \partial D$ of radius $r$. Suppose for all sufficiently small $r$, we are given two points $z_r$ and $z'_r$ on $\partial D$ and also on...
Let \( \rho(r) := |f(z_r) - f(z'_r)| \). Then there is a sequence \( \{r_n\} \) of radii that converges to 0, and \( \lim_{n \to \infty} \rho(r_n) = 0 \)

**Lemma 5.** Let \( z_0 \in \partial \mathbb{D} \). Then \( F \) in Theorem 3 converges to a limit as \( z \) approaches \( z_0 \) inside \( \partial \mathbb{D} \).

**Lemma 6.** The conformal map \( F \) in Theorem 3 extends to a continuous function from \( \mathbb{D} \) to \( \mathbb{P} \).

Theorem 3 tells us that we extended \( F \) onto the closure of \( \mathbb{D} \) continuously. Similar argument gives the inverse of \( F \) (denoted by \( G : P \to \mathbb{D} \)) can be extended continuously onto the closure of \( P \). To show that these continuous extension are inverses of each other, it amounts to taking a sequence \( z_n \in \mathbb{D} \) converging to \( z \in \partial \mathbb{D} \) and noting that \( G(F(z_n)) = z_n \). Take \( n \to \infty \). The continuity of \( F, G \) on the closure of their domains give \( G \circ F = \text{id} \), where \( \text{id} : \mathbb{D} \to \mathbb{D} \). Repeating a similar argument gives \( F \circ G = \text{id} \), where \( \text{id} : \mathbb{P} \to \mathbb{P} \).

### 0.4 The mapping formula

Suppose \( P \) is a polygonal region bounded by a polygon \( p \) whose vertices are ordered consecutively \( a_1, a_2, \ldots, a_n \in \mathbb{C} \) with \( n \geq 3 \). Denote \( \pi \alpha_k \) to be the interior angle of \( a_k \) and define the exterior angle \( \pi \beta_k \) with \( \alpha_k + \beta_k = 1 \). Since the exterior angles sum up to \( 2\pi \) in a polygon, we have \( \sum_{k=1}^{n} \beta_k = 2 \).

Now, we want to consider conformal mappings of the half plane \( \mathbb{H} \) to \( P \). Recall that \( w = \frac{z-i}{i+z} \) is a conformal mapping from \( \mathbb{H} \) to \( \mathbb{D} \) and also it is continuous on \( \mathbb{R} \). So from §0.3, we have a conformal map from \( \mathbb{D} \) to \( P \), which is continuous on \( \partial \mathbb{D} \). Therefore, we have a conformal map from \( \mathbb{H} \to P \) which maps \( \mathbb{R} \) to the polygon \( p \).

By the Riemann mapping theorem, we have an existence of a conformal mapping \( F : \mathbb{H} \to P \). We assume that none of the vertices of \( p \) corresponds to a point at infinity. Therefore, there are \( A_1, \ldots, A_n \in \mathbb{R} \) with \( A_1 < A_2 < \cdots < A_n \) so that \( F(A_k) = a_k \). We have that \( F \) maps \([A_k, A_{k+1}]\) to the line segment joined by \( a_k, a_{k+1} \) denoted by \([a_k, a_{k+1}]\). Also, \( (-\infty, A_1] \cup [A_n, \infty) \) is mapped into the edge \([a_n, a_1]\) where \( F \) maps \( \infty \) to some point (not equal to \( a_1 \) nor \( a_n \)) on \([a_n, a_1]\).
Theorem 7. Given $F : \mathbb{H} \to P$ a conformal map, there exist $c_1, c_2 \in \mathbb{C}$ so that

$$F(z) = c_1 S(z) + c_2$$ (1)

where $S(z)$ is given in §0.2 (1).

Proof: Let $1 < k < n$. Define $	ilde{A} := \{ w \in \mathbb{H} : A_{k-1} \leq \text{Re}(w) \leq A_{k+1} \}$. Consider $z \in \tilde{A}$. Recall that $F$ maps $[A_{k-1}, A_k]$ to the line segment $[a_{k-1}, a_k]$ on $p$ and $[A_k, A_{k+1}]$ to the line segment $[a_k, a_{k+1}]$ on $p$. These two line segments intersect at $a_k = F(A_k)$ at an angle $\pi \alpha_k$ (see Figure 2).

By picking a branch of the logarithm, we define

$$h_k(z) = (F(z) - a_k)^{1/\alpha_k}$$ (1.1)

where $z \in \tilde{A}$. Recall that $F$ is continuous on the real line, so that $h_k$ is continuous on $[A_{k-1}, A_{k+1}]$. By construction of $h_k$, angle between two line segments $[a_{k-1}, a_k]$, $[a_k, a_{k+1}]$ is $\alpha_k \pi$, and definition of logarithm, we have $h_k([A_{k-1}, A_{k+1}])$ is a straight line segment $L_k$ in the complex plane with $h(A_k) = 0$. We can apply the Schwarz reflection principle (scaling $h_k$ by a rotation $e^{i\theta}$, we have $e^{i\theta}h_k([A_{k-1}, A_{k+1}]) \in \mathbb{R}$ for some angle $\theta$), so $h_k$ can be analytically continued to a holomorphic function on the infinite strip $A_{k-1} < z < A_{k+1}$.
We claim that \( h'_k(z) \neq 0 \) on this infinite strip. If \( z \in \tilde{A} \), then through a computation, we have

\[
\frac{F''(z)}{F'(z) - F(A_k)} = \frac{h'_k(z)}{h_k(z)}
\]

(1.2)

and by hypothesis \( F \) is conformal on \( \mathbb{H} \), we have \( F'(z) \neq 0 \) so \( h'_k(z) \neq 0 \). By the Schwarz Reflection Principle and the proof of Schwarz Reflection Principle, we have this holds in the lower half-strip.

Now we argue that \( h'_k(z) \neq 0 \) for \( z \in (A_{k-1}, A_{k+1}) \). We invoke proposition 1.1 page 206 in [1] to do this. Notice that the image of the small half disc centered at \( z \in [A_{k-1}, A_{k+1}] \) and contained in \( \mathbb{H} \) under \( h_k \) lie on one side of \( L_k \). We know that \( h_k \) is injective on this small half disc since \( F \) is conformal. So the image of the small half disc centered at \( z \in [A_{k-1}, A_{k+1}] \) and in the lower half plane under \( h_k \) (via Schwarz Reflection principle) lie on the opposite of \( L_k \). Therefore, we have \( h'_k(z) \neq 0 \) for \( z \in (A_{k-1}, A_{k+1}) \). Hence, \( h'_k(z) \neq 0 \) for all \( z \) in the infinite strip \( A_{k-1} < \text{Re}(z) < A_{k+1} \).

Through a simple calculation, we have \( F' = \alpha_k h_k^{-\beta_k} h'_k \) and \( F'' = -\beta_k \alpha_k h_k^{-\beta_k - 1} (h'_k)^2 + \alpha_k h_k^{-\beta_k} h''_k \). We know that \( h \) has a zero of order 1 at \( A_k \) since \( h'_k(z) \neq 0 \) in the infinite strip, so we have

\[
\frac{F''(z)}{F'(z)} = \frac{-\beta_k}{z - A_k} + E_k(z),
\]

(1.3)

where \( E_k \) is holomorphic in the infinite strip. A similar result holds for \( k = 1, k = 2 \):
\[
\frac{F''(z)}{F'(z)} = -\frac{\beta_1}{z-A_1} + E_1(z) \tag{1.4}
\]

where \(E_1(z)\) is holomorphic in the infinite strip \(-\infty < \text{Re}(z) < A_2\), and

\[
\frac{F''(z)}{F'(z)} = -\frac{\beta_n}{z-A_n} + E_n(z) \tag{1.5}
\]

where \(E_n(z)\) is holomorphic in the infinite strip \(A_{n-1} < \text{Re}(z) < \infty\).

So far, we have developed that \(F\) is holomorphic on the infinite strips \(A_{k-1} < \text{Re}(z) < A_{k+1}, -\infty < \text{Re}(z) < A_2\) and \(A_{n-1} < \text{Re}(z) < \infty\). In short, we use Schwarz Reflection Principle to analytically continue \(F\) on the exterior of a disc \(|z| \leq R\) (where \(R > \max_{1 \leq k \leq n} |A_k|\)). We can also use the Schwarz principle to extend \(F\) across the segments \((-\infty, A_1), (A_n, \infty)\) since the image of these segments under \(F\) is a line segment. The fact that \(F\) maps \(\mathbb{H}\) to a bounded region shows that the analytic continuation of \(F\) outside a large disc is also bounded. Hence, holomorphic at \(\infty\) by Riemann’s theorem on removable singularities. Thus \(F''/F'\) is holomorphic at \(\infty\) and we claim that it goes to 0 as \(|z| \to \infty\).

We can expand \(F\) at \(z = \infty\) as

\[
F(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \tag{1.6}
\]

valid for large \(|z|\).

Term by term differentiation shows that \(F''/F'\) decays like \(1/z\) as \(|z|\) becomes large, which proves our claim.

Since the infinite strips overlap and cover \(\mathbb{C}\), we have

\[
\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z-A_k} \tag{1.7}
\]

is holomorphic on \(\mathbb{C}\) and is 0 at \(\infty\). By Liouville’s theorem, the quantity in (1.7) is zero. Hence,

\[
\frac{F''(z)}{F'(z)} = -\sum_{k=1}^{n} \frac{\beta_k}{z-A_k} \tag{1.8}
\]

From this, we argue that \(F'(z) = c(z-A_1)^{-\beta_1} \cdots (z-A_n)^{-\beta_n}\). Denote this product by \(Q(z)\). By taking logarithmic derivatives, we have
\[
\frac{Q'(z)}{Q(z)} = -\sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \quad (1.9)
\]

We have

\[
\frac{d}{dz} \left( \frac{F'(z)}{Q(z)} \right) = \frac{F''Q - F'Q'}{Q^2} = \frac{F'' - F' \frac{Q'}{Q}}{Q} = \frac{F'}{Q} \left( F' - \frac{Q'}{Q} \right) = 0 \quad (1.10)
\]

where the last line follows by (1.8) and (1.9). Therefore, \( F'(z) = C_1Q(z) \). Integrating yields the desired result. \( \square \)

We restate the theorem given in the introduction and provide a proof.

**Theorem 8.** If \( F \) is a conformal map from \( \mathbb{H} \) to the polygonal region \( P \) and maps the points \( A_1, \ldots, A_{n-1}, \infty \) to the vertices of \( p \), then there exist \( C_1, C_2 \in \mathbb{C} \) so that

\[
F(z) = C_1 \int_{0}^{z} \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2 \quad (2)
\]

**Proof:** By translation, we may assume \( A_j \neq 0 \) for \( j = 1, \ldots, n-1 \). Let \( A^*_n > 0 \). Consider the fractional linear transformation

\[
\Phi(z) = A^*_n - \frac{1}{z} \quad (3)
\]

See [1] (Theorem 2.4 on page 222), we have that \( \Phi \) is an automorphism of \( \mathbb{H} \). Let \( A^*_k = \Phi(A_k) \) for \( k = 1, 2, \ldots, n-1 \), and we see that \( A^*_n = \Phi(\infty) \). Since \( F(A_k) = a_k \), we have

\[
(F \circ \Phi^{-1})(A^*_k) = a_k \quad \text{for all } k = 1, 2, \ldots, n \quad (4)
\]

We can apply Theorem 7 to find that

\[
(F \circ \Phi^{-1})(z') = C_1 \int_{0}^{z'} \frac{d\zeta}{(\zeta - A^*_1)^{\beta_1} \cdots (\zeta - A^*_n)^{\beta_n}} + C_2 \quad (5)
\]
Using the substitution \( \zeta = \Phi(w) \), with differential \( d\zeta = \frac{dw}{w^2} \), and \( \sum_{k=1}^{n} \beta_k = 2 \) we have

\[
(F \circ \Phi^{-1})(z') = C_1 \int_{\Phi^{-1}(0)}^{\Phi^{-1}(z')} \frac{dw}{(A_n^* - \frac{1}{w} - A_1^*)^{\beta_1} \cdots (A_n^* - \frac{1}{w} - A_n^*)^{\beta_n} w^2} + C_2
\]

\[
= C_1 \int_{\Phi^{-1}(0)}^{\Phi^{-1}(z')} \frac{dw}{(w(A_n^* - A_1^*) - 1)^{\beta_1} \cdots (w(A_n^* - A_{n-1}^*) - 1)^{\beta_{n-1}}} + C_2
\]

\[
= C_1' \int_{0}^{\Phi^{-1}(z')} \frac{dw}{(w - 1/(A_n^* - A_1^*))^{\beta_1} \cdots (w - 1/(A_n^* - A_{n-1}^*))^{\beta_{n-1}}} + C_2'
\]

By definition of \( \Phi \), we have \( A_k = \frac{1}{A_n^* - A_k^*} \) and let \( z = \Phi^{-1}(z') \) in the previous equation and we get

\[
F(z) = C_1' \int_{0}^{z} \frac{dw}{(w - A_1^*)^{\beta_1} \cdots (w - A_{n-1}^*)^{\beta_{n-1}}} + C_2',
\]

which proves our claim.

\[\square\]

### 0.5 Matlab Implementation

#### 0.5.1 Schwarz Christoffel (SC) toolbox

We point users to the website [www.math.udel.edu/~driscoll/SC/](http://www.math.udel.edu/~driscoll/SC/) to retrieve the Matlab Schwarz Christoffel toolbox. Unpack the folder to your current working Matlab directory and open that folder.

#### 0.5.2 Matlab SC Algorithm

Given a polygon \( p \) with vertices \( a_1, \ldots, a_n \) in \( \mathbb{C} \), with interior angles \( \alpha_k \pi \) (\( \beta_k \pi \) exterior angles) and \( A_1, \ldots, A_{n-1}, A_n = \infty \) in Theorem 8 of §0.4 (call them prevertices). Recall the Schwarz-Christoffel formula for the map \( F \) is given in Theorem 8 of §0.4.

According to [2], the main practical difficulty with this formula is that, with the exception in special cases, the prevertices \( A_i \) can’t be analytically computed. [2] mentions that once three of \( A_i \)'s are chosen, with one being \( A_n \), the remaining \( n-3 \) prevertices are determined uniquely and can be obtained by solving a system of nonlinear equations. This is known as Schwarz-Christoffel parameter problem. Once the parameters are solved, the constant \( C_1 \) in Theorem 8 §0.4 can be found, and \( F, F^{-1} \) can be numerically solved.
0.5.3 A Numerical Example: Schwarz Christoffel formula (Conformal map) from \( \mathbb{H} \) to \( P \)

Consider a polygon \( P \) with vertices \( 1+i, -1+i, -1-i, 1-i \) (see image below). The toolbox computes the images of curves in \( \mathbb{H} \) under the conformal map and plots them (see plot below). Note that curves in \( \mathbb{H} \) which intersect orthogonally corresponds to curves intersecting orthogonally in \( P \). Here is the Matlab code:

```
1 p = polygon([1+i -1+i -1-i 1-i]);
2 f = hplmap(p);
3 axis([-1.5 1.5 -1.5 1.5]), hold on
4 plot(f);
```

Typing \( f \) in the command window gives the prevertices \( -1, 0, 1, \infty \) (see Figure 4 below). SC toolbox fixes these three prevertices \( -1, 1, \infty \) as three default prevertices and determines the remaining prevertex \( 0 \). Note we have the values of \( \alpha_k \) given in Figure 4. Try these commands to get values of \( F \) at these prevertices:

\[ f(-1) \]
\[ f(0) \]
\[ f(1) \]
\[ f(\infty) \]

Try these commands to get several values of \( F^{-1} \) at these vertices:

\[ \text{evalinv}(f, 1+i) \]
\[ \text{evalinv}(f, -1+i) \]
\[ \text{evalinv}(f, -1-i) \]
Another Numerical Example: Schwarz Christoffel formula (Conformal map) from \( \mathbb{D} \) to \( P \)

There is another variant of the Schwarz Christoffel formula given by a conformal map from \( \mathbb{D} \) to a polygon \( P \). The conformal \( F : \mathbb{D} \to P \) is similar to Theorem 8 in §0.4 except we have the following:

\[
F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \ldots (\zeta - A_n)^{\beta_n}} + C_2
\]  

(1)

where \( A_i \neq \infty \) for \( 1 \leq i \leq n \), \( A_i \in \partial \mathbb{D} \).

We now consider a polygon with vertices \( i, -1+i, -1-i, 1-i, 1, 0 \). The Matlab code below generates Figure 5 and Figure 6:

```matlab
1 p = polygon([i -1+i -1-i 1-i 10]);
2 plot(p);
3 f = diskmap(p)
4 plot(f)
```
The image of the origin under $F$ is called the conformal center, and mapping ten evenly spaced circles centered at $0 \in \mathbb{D}$ gives the distorted circles in Figure 5.

References
