Anthony Nguyen
Math 205B
Final Paper

The Schwarz-Christoffel Transformation

0.1 Introduction

We consider a simple polygon and ask whether there is a conformal map from $\mathbb{H}$ to the polygon denoted by P. The answer is yes and such conformal map is written in terms of a 'Schwarz-Christoffel' integral. At the end, we will include Matlab Implementation and plots.

First, we define 'Schwarz-Christoffel' integral and build up to the theorem on defining a conformal map F which maps $\mathbb{H}$ conformally to P. We state the theorem here:

Theorem 1. *If $F : \mathbb{H} \to P$ is a conformal map from the upper half-plane to the polygonal region P and maps the points $A_1, \dots, A_{n-1}, \infty \in \bar{\mathbb{R}}$ to the vertices of a polygon $\mathfrak{p}$, then there exists C_1 and C_2 such that*

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2 \tag{1}$$

where we will make these notation precise in the forth coming discussion.

0.2 The Schwarz-Christoffel Integral

We define the **Schwarz-Christoffel integral** by

$$S(z) = \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}} \tag{1}$$

where $A_1 < A_2 < \cdots < A_n$ and we assume the exponents satisfy $\beta_k < 1$ for all k with $1 < \sum_{k=1}^n \beta_k$.

Now, we make sense of the integrand in (1). We define $(z - A_k)^{\beta_k}$ the branch (defined in the complex plane slit along the infinite ray $\{A_k + iy : y \leq 0\}$) which is positive when $z = x \in (A_k, \infty)$. Therefore,

$$(z - A_k)^{\beta_k} = \begin{cases} (x - A_k)^{\beta_k} & \text{if } x \text{ is real and } x > A_k, \\ |x - A_k|^{\beta_k} e^{i\pi\beta_k} & \text{if } x \text{ is real and } x < A_k, \end{cases} \tag{2}$$

By exercise 19 in [1], the complex plane slit along the union of the rays $\bigcup_{k=1}^n \{A_k + iy : y \leq 0\}$ (denote it by Ω) is simply connected. Therefore, $S(z)$ is holomorphic on Ω. Since

$\beta_k < 1$, this implies that we can integrate $(\zeta - A_k)^{-\beta_k}$ around A_k for $k = 1, \ldots, n$. This means that S is continuous up to the real line, including A_k. We have that S can be integrated along any path that avoids the open slits $\bigcup_{k=1}^{n}\{A_k + iy : y < 0\}$.

For large $|\zeta|$, there exists some positive constant c so that

$$\left|\prod_{k=1}^{n}(\zeta - A_k)^{-\beta_k}\right| \leq c|\zeta|^{-\sum \beta_k} \tag{3}$$

Given $\sum_{k=1}^{n} \beta_k > 1$, this implies for sufficiently large $|z|$, we have

$$\int_{z}^{\infty} \frac{1}{|\zeta|^{\sum \beta_k}} d\zeta \tag{4}$$

exists and is finite. To see this, integrating $|\zeta|^{-\sum \beta_k}$ along a path starting at iy with $y \gg 1$ to $i\infty$ exists and is finite (by the p-test). Using this fact and Cauchy's theorem imply that $\lim_{r\to+\infty} S(re^{i\theta})$ exists and is independent of the angle $\theta \in [0, 2\pi]$ (call this limit a_∞). We let $a_k := S(A_k)$ for $k = 1, \ldots, n$.

We introduce a proposition that says that S maps $\mathbb{R}$ onto the edges that bound a polygon whose vertices are given by $a_1, \ldots, a_n$.

Theorem 2. *Suppose $S(z)$ is given by (1).*

(i) If $\sum_{k=1}^{n} \beta_k = 2$, and $\mathfrak{p}$ denotes the polygon whose vertices are given (in order) by $a_1, \ldots, a_n$, then S maps the real axis onto $\mathfrak{p} - \{a_\infty\}$ (meaning $\mathbb{R}$ gets mapped into the edges of the polygon). The point a_∞ lies on the segment $[a_n, a_1]$ and is the image of the point at infinity. Also, the interior angle at the vertex a_k is $\alpha_k\pi$ where $\alpha_k = 1 - \beta_k$.

(ii) There is a similar conclusion when $1 < \sum_{k=1}^{n} \beta_k < 2$, except now the image of the extended line $\overline{\mathbb{R}}$ is the polygon of $n+1$ sides with vertices $a_1, a_2, \ldots, a_n, a_\infty$. The angle at the vertex a_∞ is $\alpha_\infty\pi$ where $\alpha_\infty = 1 - \beta_\infty$, where $\beta_\infty := 2 - \sum_{k=1}^{n} \beta_k$.

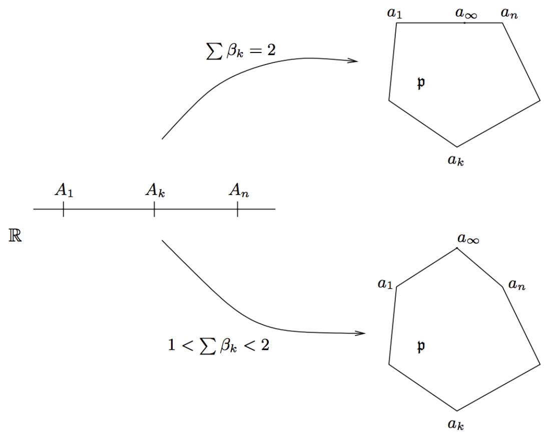


Figure 1: Action of the integral $S(z)$

Remember we are trying to find a conformal map from $\mathbb{H}$ to a given region P that is bounded by a polygon, and the above theorem does not address this. We give two reasons for this.

- It is not true for general n and generic choices of $A_1, \ldots, A_n$ that the image of S under $\overline{\mathbb{R}}$ is a simple polygon. Also it is not generally true that the mapping S is conformal on $\mathbb{H}$.
- The theorem above does not show that starting with a simply connected region P whose boundary is a polygon $\mathfrak{p}$, the mapping S is a conformal map from $\mathbb{H}$ to P for certain choices of $A_1, \ldots, A_n$ and simple modifications. It turns out this is true and we will discuss this further.

0.3 Boundary Behavior

In this section, we consider a polygonal region P, a bounded simply connected open set whose boundary is a polygonal line $\mathfrak{p}$.

We like to study conformal maps from the half-plane $\mathbb{H}$ to P, so we consider studying conformal maps from the disk $\mathbb{D}$ to P (and their behaviors on $\partial\mathbb{D}$). We introduce a theorem that discusses this.

Theorem 3. *If $F : \mathbb{D} \to P$ is a conformal map, then F extends to a continuous bijection from the closure $\overline{\mathbb{D}}$ of the disk $\mathbb{D}$ to the closure $\overline{P}$ of the polygonal region. More specifically, F is a bijection from $\partial\mathbb{D}$ to the boundary polygon $\mathfrak{p}$.*

The idea is to show that if $z_0 \in \partial\mathbb{D}$, then $\lim_{z\to z_0} F(z)$ exists. We introduce three (3) lemmas (each depends on the previous lemma). See [1] for more detail on the proof. The first lemma we introduce, we assume $f : \mathbb{D} \to \mathbb{C}$ is conformal.

Lemma 4. *For $r \in (0, \frac{1}{2})$, denote C_r by the circle centered at $z_0 \in \partial\mathbb{D}$ of radius r. Suppose for all sufficiently small r, we are given two points z_r and z_r' on $\partial\mathbb{D}$ and also on*

C_r. *Let* $\rho(r) := |f(z_r) - f(z_r')|$. *Then there is a sequence* $\{r_n\}$ *of radii that converges to* 0*, and* $\lim_{n\to\infty} \rho(r_n) = 0$

Lemma 5. *Let* $z_0 \in \partial\mathbb{D}$. *Then* F *in Theorem* 3 *converges to a limit as* z *approaches* z_0 *inside* $\partial\mathbb{D}$.

Lemma 6. *The conformal map* F *in Theorem* 3 *extends to a continuous function from* $\overline{\mathbb{D}}$ *to* $\overline{P}$.

Theorem 3 tells us that we extended F onto the closure of $\mathbb{D}$ continuously. Similar argument gives the inverse of F (denoted by $G : P \to \mathbb{D}$) can be extended continuously onto the closure of P. To show that these continuous extension are inverses of each other, it amounts to taking a sequence $z_n \in \mathbb{D}$ converging to $z \in \partial\mathbb{D}$ and noting that $\mathrm{G}(\mathrm{F}(\mathrm{z}_n)) = z_n$. Take $n \to \infty$. The continuity of F, G on the closure of their domains give $G \circ F = \mathrm{id}$, where $\mathrm{id} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$. Repeating a similar argument gives $F \circ G = \mathrm{id}$, where $\mathrm{id} : \overline{P} \to \overline{P}$.

0.4 The mapping formula

Suppose P is a polygonal region bounded by a polygon $\mathfrak{p}$ whose vertices are ordered consecutively $a_1, a_2, \ldots, a_n \in \mathbb{C}$ with $n \geq 3$. Denote $\pi\alpha_k$ to be the interior angle of a_k and define the exterior angle $\pi\beta_k$ with $\alpha_k + \beta_k = 1$. Since the exterior angles sum up to 2π in a polygon, we have $\sum_{k=1}^{n} \beta_k = 2$.

Now, we want to consider conformal mappings of the half plane $\mathbb{H}$ to P. Recall that $w = \frac{i-z}{i+z}$ is a conformal mapping from $\mathbb{H}$ to $\mathbb{D}$ and also it is continuous on $\mathbb{R}$. So from §0.3, we have a conformal map from $\mathbb{D}$ to P, which is continuous on $\partial\mathbb{D}$. Therefore, we have a conformal map from $\mathbb{H} \to P$ which maps $\mathbb{R}$ to the polygon $\mathfrak{p}$.

By the Riemann mapping theorem, we have an existence of a conformal mapping $F : \mathbb{H} \to P$. We assume that none of the vertices of $\mathfrak{p}$ corresponds to a point at infinity. Therefore, there are $A_1, \ldots, A_n \in \mathbb{R}$ with $A_1 < A_2 < \cdots < A_n$ so that $F(A_k) = a_k$. We have that F maps $[A_k, A_{k+1}]$ to the line segment joined by a_k, a_{k+1} denoted by $[a_k, a_{k+1}]$. Also, $(-\infty, A_1] \cup [A_n, \infty)$ is mapped into the edge $[a_n, a_1]$ where F maps ∞ to some point (not equal to a_1 nor a_n) on $[a_n, a_1]$.

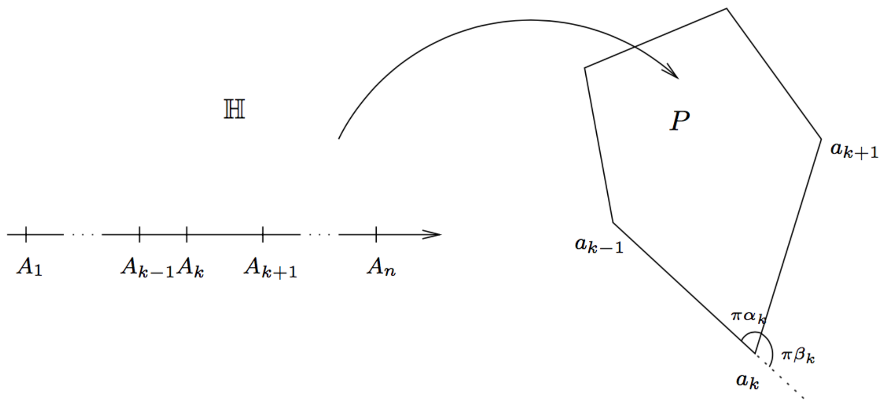


Figure 2: The mapping F

Theorem 7. *Given $F : \mathbb{H} \to P$ a conformal map, there exist $c_1, c_2 \in \mathbb{C}$ so that*

$$F(z) = c_1 S(z) + c_2 \tag{1}$$

where $S(z)$ is given in §0.2 (1).

Proof: *Let $1 < k < n$. Define $\tilde{A} := \{w \in \mathbb{H} : A_{k-1} \leq Re(w) \leq A_{k+1}\}$. Consider $z \in \tilde{A}$. Recall that F maps $[A_{k-1}, A_k]$ to the line segment $[a_{k-1}, a_k]$ on $\mathfrak{p}$ and $[A_k, A_{k+1}]$ to the line segment $[a_k, a_{k+1}]$ on $\mathfrak{p}$. These two line segments intersect at $a_k = F(A_k)$ at an angle $\pi\alpha_k$ (see Figure 2).*

By picking a branch of the logarithm, we define

$$h_k(z) = (F(z) - a_k)^{1/\alpha_k} \tag{1.1}$$

where $z \in \tilde{A}$. Recall that F is continuous on the real line, so that h_k is continuous on $[A_{k-1}, A_{k+1}]$. By construction of h_k, angle between two line segments $[a_{k-1}, a_k]$, $[a_k, a_{k+1}]$ is $\alpha_k\pi$, and definition of logarithm, we have $h_k([A_{k-1}, A_{k+1}])$ is a straight line segment L_k in the complex plane with $h(A_k) = 0$. We can apply the Schwarz reflection principle (scaling h_k by a rotation $e^{i\theta}$, we have $e^{i\theta}h_k([A_{k-1}, A_{k+1}]) \in \mathbb{R}$ for some angle θ), so h_k can be analytically continued to a holomorphic function on the infinite strip $A_{k-1} < z < A_{k+1}$

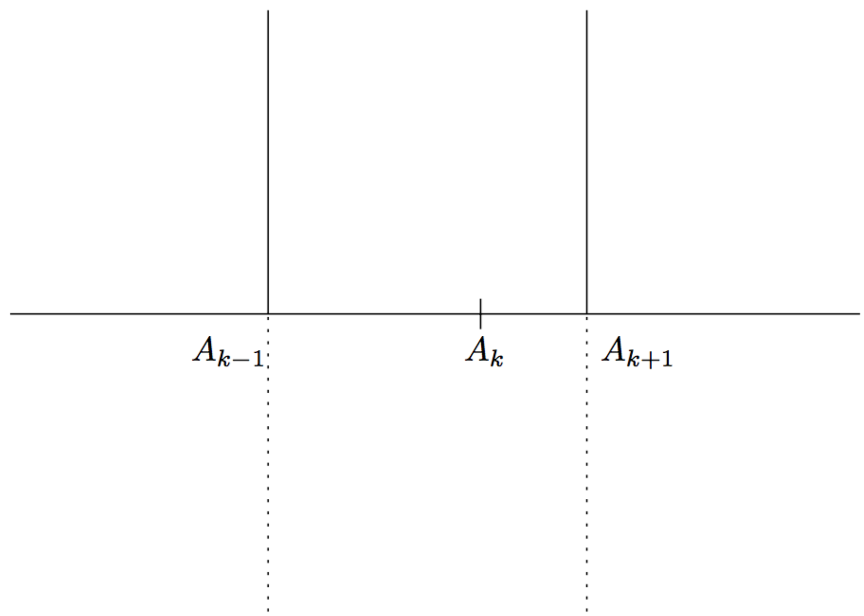


Figure 3: Schwarz Reflection

We claim that $h_k'(z) \neq 0$ *on this infinite strip. If* $z \in \tilde{A}$, *then through a computation, we have*

$$\frac{F'(z)}{F(z) - F(A_k)} = \alpha_k \frac{h_k'(z)}{h_k(z)} \tag{1.2}$$

and by hypothesis F *is conformal on* $\mathbb{H}$, *we have* $F'(z) \neq 0$ *so* $h_k'(z) \neq 0$. *By the Schwarz Reflection Principle and the proof of Schwarz Reflection Principle, we have this holds in the lower half-strip.*

Now we argue that $h_k'(z) \neq 0$ *for* $z \in (A_{k-1}, A_{k+1})$. *We invoke proposition* 1.1 *page* 206 *in* [1] *to do this. Notice that the image of the small half disc centered at* $z \in [A_{k-1}, A_{k+1}]$ *and contained in* $\mathbb{H}$ *under* h_k *lie on one side of* L_k. *We know that* h_k *is injective on this small half disc since* F *is conformal. So the image of the small half disc centered at* $z \in [A_{k-1}, A_{k+1}]$ *and in the lower half plane under* h_k *(via Schwarz Reflection principle) lie on the opposite of* L_k. *Therefore, we have* $h_k'(z) \neq 0$ *for* $z \in (A_{k-1}, A_{k+1})$. *Hence,* $h_k'(z) \neq 0$ *for all* z *in the infinite strip* $A_{k-1} < Re(z) < A_{k+1}$.

Through a simple calculation, we have $F' = \alpha_k h_k^{-\beta_k} h_k'$ *and* $F'' = -\beta_k \alpha_k h_k^{-\beta_k - 1}(h_k')^2 + \alpha_k h_k^{-\beta_k} h_k''$. *We know that* h *has a zero of order* 1 *at* A_k *since* $h_k'(z) \neq 0$ *in the infinite strip, so we have*

$$\frac{F''(z)}{F'(z)} = \frac{-\beta_k}{z - A_k} + E_k(z), \tag{1.3}$$

where E_k *is holomorphic in the infinite strip. A similar result holds for* $k = 1, k = 2$*:*

$$\frac{F''(z)}{F'(z)} = -\frac{\beta_1}{z - A_1} + E_1(z) \tag{1.4}$$

where $E_1(z)$ *is holomorphic in the infinite strip* $-\infty < Re(z) < A_2$*, and*

$$\frac{F''(z)}{F'(z)} = -\frac{\beta_n}{z - A_n} + E_n(z) \tag{1.5}$$

where $E_n(z)$ *is holomorphic in the infinite strip* $A_{n-1} < Re(z) < \infty$*.*

So far, we have developed that F *is holomorphic on the infinite strips* $A_{k-1} < Re(z) < A_{k+1}$*,* $-\infty < Re(z) < A_2$ *and* $A_{n-1} < Re(z) < \infty$*. In short, we use Schwarz Reflection Principle to analytically continue* F *on the exterior of a disc* $|z| \leq R$ *(where* $R > \max_{1 \leq k \leq n} |A_k|$*). We can also use the Schwarz principle to extend* F *across the segments* $(-\infty, A_1), (A_n, \infty)$ *since the image of these segments under* F *is a line segment. The fact that* F *maps* $\mathbb{H}$ *to a bounded region shows that the analytic continuation of* F *outside a large disc is also bounded. Hence, holomorphic at* ∞ *by Riemann's theorem on removable singularities. Thus* F''/F' *is holomorphic at* ∞ *and we claim that it goes to* 0 *as* $|z| \to \infty$*. We can expand* F *at* $z = \infty$ *as*

$$F(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \tag{1.6}$$

valid for large $|z|$*.*

Term by term differentiation shows that F''/F' *decays like* $1/z$ *as* $|z|$ *becomes large, which proves our claim.*

Since the infinite strips overlap and cover $\mathbb{C}$*, we have*

$$\frac{F''(z)}{F'(z)} + \sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \tag{1.7}$$

is holomorphic on $\mathbb{C}$ *and is* 0 *at* ∞*. By Liouville's theorem, the quantity in* (1.7) *is zero. Hence,*

$$\frac{F''(z)}{F'(z)} = -\sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \tag{1.8}$$

From this, we argue that $F'(z) = c(z - A_1)^{-\beta_1} \cdots (z - A_n)^{-\beta_n}$*. Denote this product by* $Q(z)$*. By taking logarithmic derivatives, we have*

$$\frac{Q'(z)}{Q(z)} = -\sum_{k=1}^{n} \frac{\beta_k}{z - A_k} \tag{1.9}$$

We have

$$\begin{aligned}\frac{d}{dz}\left(\frac{F'(z)}{Q(z)}\right) &= \frac{F''Q - F'Q'}{Q^2} \\ &= \frac{F''}{Q} - \frac{F'}{Q}\frac{Q'}{Q} \\ &= \frac{F'}{Q}\left(\frac{F''}{F'} - \frac{Q'}{Q}\right) \\ &= 0 \end{aligned} \tag{1.10}$$

where the last line follows by (1.8) *and* (1.9). *Therefore,* $F'(z) = C_1 Q(z)$. *Integrating yields the desired result.* □

We restate the theorem given in the introduction and provide a proof.

Theorem 8. *If F is a conformal map from $\mathbb{H}$ to the polygonal region P and maps the points $A_1, \dots, A_{n-1}, \infty$ to the vertices of $\mathfrak{p}$, then there exist $C_1, C_2 \in \mathbb{C}$ so that*

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_{n-1})^{\beta_{n-1}}} + C_2 \tag{2}$$

Proof: *By translation, we may assume $A_j \neq 0$ for $j = 1, \dots, n-1$. Let $A_n^* > 0$. Consider the fractional linear transformation*

$$\Phi(z) = A_n^* - \frac{1}{z} \tag{3}$$

See [1] *(Theorem* 2.4 *on page* 222*), we have that Φ is an automorphism of $\mathbb{H}$. Let $A_k^* = \Phi(A_k)$ for $k = 1, 2, \dots, n-1$, and we see that $A_n^* = \Phi(\infty)$. Since $F(A_k) = a_k$, we have*

$$(F \circ \Phi^{-1})(A_k^*) = a_k \quad \textit{for all } k = 1, 2, \dots, n \tag{4}$$

We can apply Theorem 7 *to find that*

$$(F \circ \Phi^{-1})(z') = C_1 \int_0^{z'} \frac{d\zeta}{(\zeta - A_1^*)^{\beta_1} \cdots (\zeta - A_n^*)^{\beta_n}} + C_2 \tag{5}$$

Using the substitution $\zeta = \Phi(w)$, with differential $d\zeta = \frac{dw}{w^2}$, and $\sum_{k=1}^{n} \beta_k = 2$ we have

$$
\begin{aligned}
(F \circ \Phi^{-1})(z') &= C_1 \int_{\Phi^{-1}(0)}^{\Phi^{-1}(z')} \frac{dw}{(A_n^* - \frac{1}{w} - A_1^*)^{\beta_1} \cdots (A_n^* - \frac{1}{w} - A_n^*)^{\beta_n} w^2} + C_2 \\
&= C_1 \int_{\Phi^{-1}(0)}^{\Phi^{-1}(z')} \frac{dw}{(w(A_n^* - A_1^*) - 1)^{\beta_1} \cdots (w(A_n^* - A_{n-1}^*) - 1)^{\beta_{n-1}}} + C_2 \\
&= C_1 \int_{0}^{\Phi^{-1}(z')} \frac{dw}{(w(A_n^* - A_1^*) - 1)^{\beta_1} \cdots (w(A_n^* - A_{n-1}^*) - 1)^{\beta_{n-1}}} + C_2' \\
&= C_1' \int_{0}^{\Phi^{-1}(z')} \frac{dw}{(w - 1/(A_n^* - A_1^*))^{\beta_1} \cdots (w - 1/(A_n^* - A_{n-1}^*))^{\beta_{n-1}}} + C_2' \quad (6)
\end{aligned}
$$

By definition of Φ, we have $A_k = \frac{1}{A_n^* - A_k^*}$ and let $z = \Phi^{-1}(z')$ in the previous equation and we get

$$
F(z) = C_1' \int_0^z \frac{dw}{(w - A_1)^{\beta_1} \cdots (w - A_{n-1})^{\beta_{n-1}}} + C_2', \quad (7)
$$

which proves our claim. □

0.5 Matlab Implementation

0.5.1 Schwarz Christoffel (SC) toolbox

We point users to the website www.math.udel.edu/~driscoll/SC/ to retrieve the Matlab Schwarz Christoffel toolbox. Unpack the folder to your current working Matlab directory and open that folder.

0.5.2 Matlab SC Algorithm

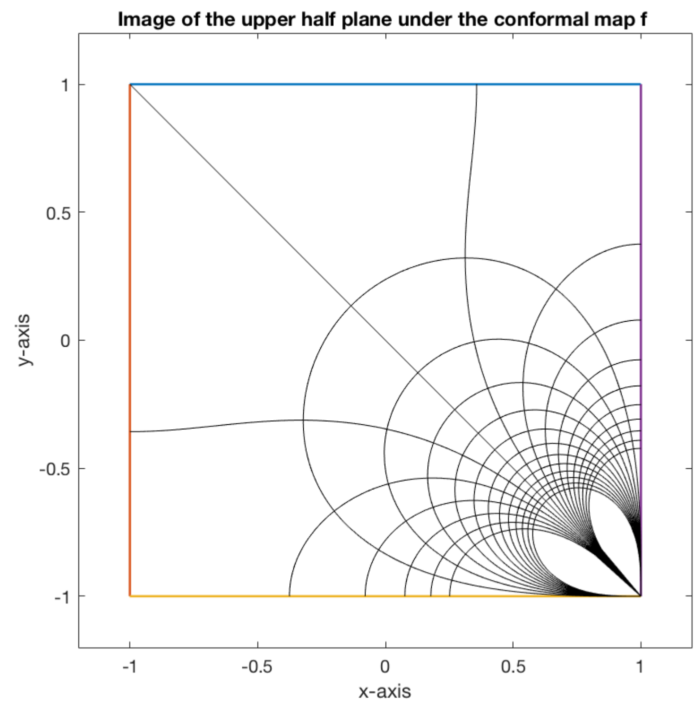
*Given a polygon $\mathfrak{p}$ with vertices $a_1, \ldots, a_n$ in $\mathbb{C}$, with interior angles $\alpha_k \pi$ ($\beta_k \pi$ exterior angles) and $A_1, \ldots, A_{n-1}, A_n = \infty$ in Theorem 8 of §0.4 (call them **prevertices**). Recall the Schwarz-Christoffel formula for the map F is given in Theorem 8 of §0.4.*

*According to [2], the main practical difficulty with this formula is that, with the exception in special cases, the prevertices A_i can't be analytically computed. [2] mentions that once three of A_i's are chosen, with one being A_n, the remaining $n-3$ prevertices are determined uniquely and can be obtained by solving a system of nonlinear equations. This is known as **Schwarz-Christoffel parameter problem**. Once the parameters are solved, the constant C_1 in Theorem 8 §0.4 can be found, and F, F^{-1} can be numerically solved.*

0.5.3 A Numerical Example: Schwarz Christoffel formula (Conformal map) from $\mathbb{H}$ to P

Consider a polygon P with vertices $1+i, -1+i, -1-i, 1-i$ (see image below). The tool box computes the images of curves in $\mathbb{H}$ under the conformal map and plots them (see plot below). Note that curves in $\mathbb{H}$ which intersect orthogonally corresponds to curves intersecting orthogonally in P. Here is the Matlab code:

```
p = polygon([1+i -1+i -1-i 1-i]);
f = hplmap(p);
axis([-1.5 1.5 -1.5 1.5]), hold on
plot(f);
```



Typing f in the command window gives the prevertices $-1, 0, 1, \infty$ (see Figure 4 below). SC toolbox fixes these three prevertices $-1, 1, \infty$ as three default prevertices and determines the remaining prevertex 0. Note we have the values of α_k given in Figure 4. Try these commands to get values of F at these prevertices:

$\gg f(-1)$
$\gg f(0)$
$\gg f(1)$
$\gg f(\text{inf})$

Try these commands to get several values of F^{-1} at these vertices:

$\gg \text{evalinv}(f, 1+i)$
$\gg \text{evalinv}(f, -1+i)$
$\gg \text{evalinv}(f, -1-i)$

$\gg$ evalinv$(f, 1-i)$

```
>> f

 hplmap object:

     vertex                  alpha         prevertex
 --------------------------------------------------------
  1.00000 + 1.00000i      0.50000     -1.000000000000e+00
 -1.00000 + 1.00000i      0.50000      0.000000000000e+00
 -1.00000 - 1.00000i      0.50000      1.000000000000e+00
  1.00000 - 1.00000i      0.50000                     Inf

  c = 0.76275976 - 9.341113e-17i

  Apparent accuracy is 1.12e-11
```

Figure 4: Data on the conformal map F

0.5.4 Another Numerical Example: Schwarz Christoffel formula (Conformal map) from $\mathbb{D}$ to P

There is another variant of the Schwarz Christoffel formula given by a conformal map from $\mathbb{D}$ to a polygon P. The conformal $F : \mathbb{D} \to P$ is similar to Theorem 8 in §0.4 except we have the following:

$$F(z) = C_1 \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}} + C_2 \tag{1}$$

where $A_i \neq \infty$ for $1 \leq i \leq n$, $A_i \in \partial\mathbb{D}$.

We now consider a polygon with vertices $i, -1+i, -1-i, 1-i, 1, 0$. The Matlab code below generates Figure 5 and Figure 6:

```
p = polygon([i -1+i -1-i 1-i 10]);
plot(p);
f = diskmap(p)
plot(f)
```

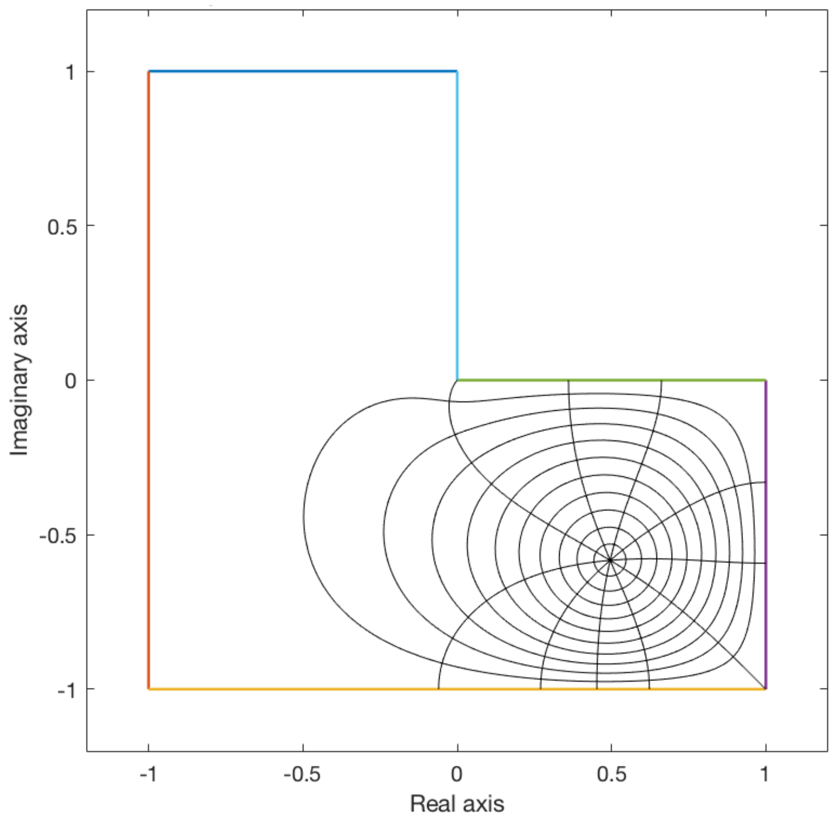


Figure 5: Image of circles under the conformal map F

```
diskmap object:

     vertex              alpha         prevertex              arg/pi
-----------------------------------------------------------------------
 0.00000 + 1.00000i     0.50000     0.98974 + 0.14286i     0.045628948204
-1.00000 + 1.00000i     0.50000     0.98811 + 0.15378i     0.049144854230
-1.00000 - 1.00000i     0.50000     0.95325 + 0.30217i     0.097710854029
 1.00000 - 1.00000i     0.50000    -1.00000 + 0.00000i     1.000000000000
 1.00000 + 0.00000i     0.50000    -0.00000 - 1.00000i     1.500000000000
 0.00000 + 0.00000i     1.50000     1.00000 + 0.00000i     2.000000000000

 c = -0.48783135 + 0.29499692i
 Conformal center at 0.4955 - 0.5829i

 Apparent accuracy is 6.31e-08
```

Figure 6: Image of circles under the conformal map F

The image of the origin under F is called the conformal center, and mapping ten evenly spaced circles centered at $0 \in \mathbb{D}$ gives the distorted circles in Figure 5.

References

[1] Stein, Elias M., and Rami Shakarchi. Princeton Lectures in Analysis: Complex Analysis. Princeton Univ. Press, 2003.

[2] Driscoll, T. A. . Schwarz - Christoffel Toolbox Users Guide. Retrieved from http://www.math.udel.edu/~driscoll/SC/guide-v5.pdf