Teichmüller Spaces

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1 Introduction

Teichmüller spaces parameterize the complex structures on fixed surfaces. In this report, we'll satisfy ourselves with only a general overview of the spaces. For a more in depth introduction to the topic, see [IT92]. We start with a comparison of two familiar spaces:

> *A **surface** is real connected two dimensional manifold.*
>
> *A **Riemann surface** is a complex connected one dimensional manifold.*

These curt definitions highlight the similarities between the two types of objects. Tucked away in the details, there are enough discrepancies to spur a closer comparison.

1.1 Surfaces vs. Riemann Surfaces

An atlas on a Riemann surface induces what is called a *complex structure* - it allows for a coherent and unambiguous definition of holomorphic maps. We'll organize our comparison into two major categories: First, we consider the effects of forgetting a complex structure. Then we exam the process of trying to add a complex structure to a surface. For now, we'll restrict our attention to closed surfaces and Riemann surfaces.

First, some vocabulary: An open subset $U \subseteq S$ of a surface (or Riemann surface) S is a *coordinate patch* if there exists a map $f : U \to \mathbb{R}^2$ (resp. $\mathbb{C}$) which is compatible with the atlas on S. A *homeomorphism* is a continuous map with a continuous inverse, a *diffeomorphism* is a differentiable map with a differentiable inverse, and a *biholomorphism* is a holomorphism with a holomorphic inverse. We'll make use of the fact that biholomorphic implies diffeomorphic, which implies homeomorphic.

1.1.1 Forgetting the Complex Structure.

Suppose that S is a Riemann surface, with complex structure induced by an atlas

$$\mathcal{A} := \Big(\{U_j \subset S\}, \{z_j : U_j \to \mathbb{C}\};\ j \in I\Big).$$

In particular, for each pair $k, j \in I$ such that $U_k \cap U_j \neq \varnothing$ the transition function

$$z_{kj} := z_k \circ z_j^{-1} :\ z_j\big(U_k \cap U_j\big) \longrightarrow z_k\big(U_k \cap U_j\big)$$

is biholomorphic. Every complex number $w \in \mathbb{C}$ can be identified with a point in $\mathbb{R}^2$, via the association $w \leftrightarrow \big(\mathrm{Re}(w), \mathrm{Im}(w)\big)$. Under this correspondence, the coordinate maps in $\mathcal{A}$ become maps to $\mathbb{R}^2$. Since they're biholomorphic by assumption, the transition maps become homeomorphisms between open subsets of $\mathbb{R}^2$. Our first observation comes as a bit of a relief - we conclude that *Riemann surfaces are surfaces.*

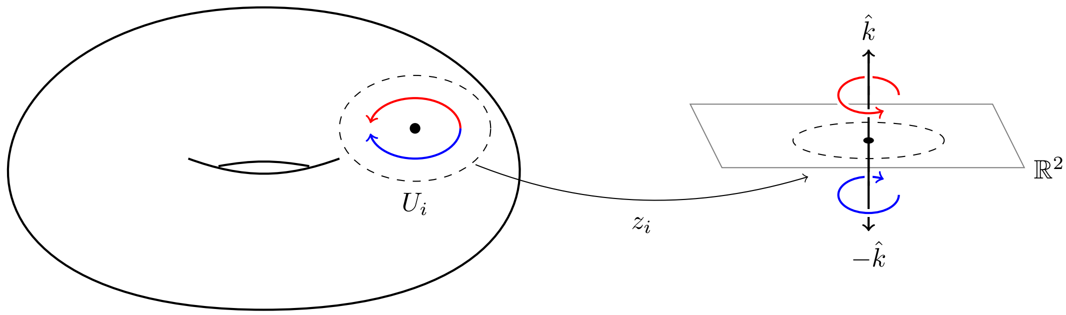

Figure 1: Each point on the surface S has two choices of orientation induced by its local coordinate patch. These choices correspond to the right hand rule and choice of normal direction in $\mathbb{R}^3$. As long as the transition maps preserve orientation (i.e. don't send $\hat{k} \to -\hat{k}$) it will be possible to make a consistent global choice of orientation.

1.1.2 Adding a Complex Structure

When is a surface a Riemann surface? If we were in a meticulous mood, we could say 'never,' since a Riemann surface has more structure than a general topological surface. (In the same way that a set is not a group, and a group is not a ring.) It turns out the answer is more nuanced than 'never' or 'always.'

Suppose that S is a surface which admits a complex structure, i.e. there's some atlas $\mathcal{A}$ over S with biholomorphic transition maps $z_{kj} : z_j(U_k \cap U_j) \longrightarrow z_k(U_k \cap U_j)$. We claim that the mere existence of this atlas implies that S is *orientable* - that is, there's some consistent global definition of 'clockwise' on the surface.

In what follows, we will identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, and use $x_{kj} := \mathrm{Re}(z_{kj})$, $y_{kj} := \mathrm{Im}(z_{kj})$ to denote the real and imaginary parts of the transition maps, so that $z_{kj} = x_{kj} + iy_{kj}$.

On each coordinate path U_j, the map $z_j : U_j \to \mathbb{R}^2$ induces two choices of orientation in U_j. Namely, if we embed $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ then these orientations correspond choice of unit normal (with orientation following the right hand rule.) As long as the transition maps preserve orientation in $\mathbb{R}^2$, we will be able to make a compatible choice of orientation in each coordinate patch.

A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves orientation exactly when its Jacobian has positive determinant. For a transformation map $z_{kj} : z_j(U_k \cap U_j) \to z_k(U_k \cap U_j)$ the relevant quantity is

$$\det J(z_{kj}) = \det \begin{pmatrix} \dfrac{\partial x_{kj}}{\partial x} & \dfrac{\partial x_{kj}}{\partial y} \\ \dfrac{\partial y_{kj}}{\partial x} & \dfrac{\partial y_{kj}}{\partial y} \end{pmatrix} = \frac{\partial x_{kj}}{\partial x}\frac{\partial y_{kj}}{\partial y} - \frac{\partial x_{kj}}{\partial y}\frac{\partial y_{kj}}{\partial x}.$$

Since the transition maps are diffeomorphisms, we know that this determinant is non-zero. By assumption, z_{kj} is holomorphic, and hence also satisfies the Cauchy-Riemann equations:

$$\frac{\partial x_{kj}}{\partial x} = \frac{\partial y_{kj}}{\partial y}; \qquad \frac{\partial x_{kj}}{\partial y} = -\frac{\partial y_{kj}}{\partial x}.$$

This implies that

$$\det J(z_{kj}) = \left(\frac{\partial x_{kj}}{\partial x}\right)^2 + \left(\frac{\partial x_{kj}}{\partial y}\right)^2 > 0.$$

This leads to our second major conclusion: *The existence of complex structure implies that a surface is orientable.*

Compact oriented surfaces are classified (up to homeomorphism!) by their genus. Suppose that Σ_g is a surface of genus g. To construct Σ_g, we start with a regular polygon with $4g$ sides laying in $\mathbb{R}^2$ then identify pairs of sides. See Figure 2 for details. The result is a surface with coordinate charts induced from

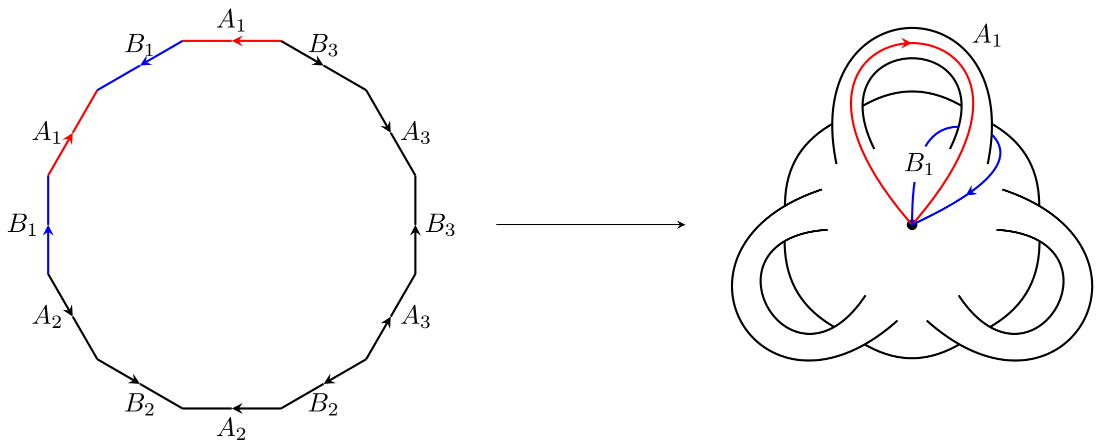

Figure 2: Building Σ_3 as a quotient of $\mathbb{R}^2$. Sides with the same label are glued together according to the directions indicated by arrows.

the identity map $\mathbb{R}^2 \to \mathbb{R}^2$. By considering the polygon as a subset of $\mathbb{C}$ instead of $\mathbb{R}^2$, the same process produces a Riemann surface.

In conclusion, *every compact, orientable surface can be made into a Riemann surface.*

1.2 Various Complex Structures

A closer look at the relevant notions of equivalence for our two types of surfaces hints that the situation is more involved than the preceding discussion would imply.

To be concrete, we'll need a few more definitions. A map $f : S \to S'$ between two (Riemann) surfaces is called a *homeomorphism* if for every pair of coordinate patches $U \subset S$ and $U' \subset S'$, with coordinate maps $z : U \to \mathbb{R}^2$ and $w : U' \to \mathbb{R}^2$ such that $f(U) \subset U'$, the composite map $w \circ f \circ z^{-1}$ is homeomorphic on its domain of definition.

Similarly, a map between Riemann surfaces is called *biholomorphic* if the corresponding definition holds, with $\mathbb{R}^2$ replaced by $\mathbb{C}$ and the composite map biholomorphic. Two (Riemann) surfaces are *homeomorphic* if there exists a homeomorphism between them, and two Riemann surfaces[1] are *biholomorphic* if there exists a biholomorphism between them.

Biholomorphic implies homeomorphic, but the converse generally is not true. So, we direct our attention at the various complex structures defined over a fixed surface. This is where the subtly lies - it turns out that, in general, there are infinitely many non-equivalent complex structures associated to each fixed surface. Teichmüller spaces parametrize this collection of structures.

2 Teichmüller Spaces: Two Perspectives.

We'll give two descriptions of Teichmller spaces and prove that they coincide.

2.1 Marked Riemann Surfaces

We start by definition the Teichmüller space of genus g, written T_g. Recall that the *fundamental group* associated to a space X, written $\pi_1(X, p)$, is generated by homotopy classes of maps $S^1 \to X$ that pass through p. For connected manifolds the choice of base point p does not affect the group's structure. The fundamental group of a Riemann surface R of genus g is generated by $2g$ elements. A set of generators $\Sigma_p = \{[A_j], [B_j] \mid j = 1, \dots, g\}$ of $\pi(R, p)$ is called a *marking* on R.

[1]Without some complex structure in place, it makes very little sense to talk about 'biholomorphic surfaces.'

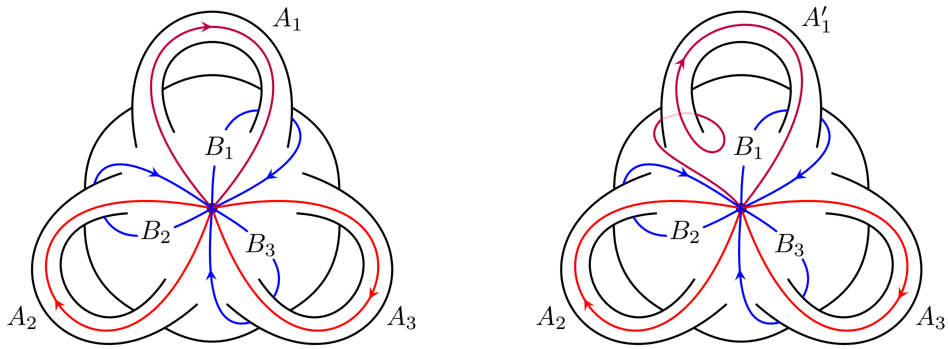

Figure 3: Collections of paths associated with the markings $\Sigma_p = \{[A_1],[A_2],[A_3],[B_1],[B_2],[B_3]\}$, and $\Sigma'_p = \{[A'_1],[A_2],[A_3],[B_1],[B_2],[B_3]\}$ on a surface of genus 3.

Two pairs (R,Σ'_p) and (S,Σ_q) of marked closed Riemann surfaces are called *equivalent* if there's some biholomorphic map $f : R \to S$ and some continuous curve Γ connecting p to $f(q)$ on S such that

$$[C'_i] = [\Gamma^{-1} \cdot (f \circ C_i) \cdot \Gamma]$$

for all C'_i, C_i in Σ'_p and Σ_p, respectively. Note, in particular, that this equivalence implies that R and S have the same genus. The equivalence class of a pair (S,Σ_p) is denoted $[S,\Sigma_p]$ and the set of all such classes is called the *Teichmller space* T_g.

2.2 Orientation Preserving Diffeomorphisms

Fix a closed Riemann surface R. The *Teichmüller space* $T(R)$ associated to R is defined as follows. Start with the set $\{(S,f)\}$ of orientation preserving diffeomorphisms $f : R \to S$, where S can be any Riemann surface. Two such diffeomorphisms $f_0 : R \to S_0$ and $f_1 : R \to S_1$ are equivalent, written $(S_0,f_0) \sim (S_1,f_1)$, if the composite map

$$f_1 \circ f_0^{-1} : S_0 \longrightarrow S_1$$

is *homotopic* to some biholomorphic map $h : S_0 \to S_1$. In other words, there's some map (called a homotopy)

$$H : [0,1] \times S_0 \longrightarrow S_1$$

Which is continuous in the first variable, such that $H(t,\cdot) : S_0 \to S_1$ is an orientation preserving diffeomorphism for each $t \in [0,1]$, $H(0,\cdot) = f_1 \circ f_0^{-1}(\cdot)$, and $H(1,\cdot) = h(\cdot)$.

Claim: $\sim$ *is an equivalence relation.*

Each (S,f) is equivalent to itself, since $f \circ f^{-1} : S \to S$ is the identity map, and hence biholomorphic. Suppose that $(S_0,f_0) \sim (S_1,f_1)$ and $(S_1,f_1) \sim (S_2,f_2)$. We want to show that $(S_0,f_0) \sim (S_2,f_2)$. Let $H_{01} : [0,1] \times S_0 \to S_1$ and $H_{12} : [0,1] \times S_1 \to S_2$ denote homotopies connecting $f_1 \circ f_0^{-1}$ and $f_2 \circ f_1^{-1}$ to biholomorphisms. Define H_{02} as follows:

$$\begin{aligned} H_{02} : [0,1] \times S_0 &\longrightarrow S_2 \\ (t,z) &\longmapsto H_{12}\big(t, H_{01}(t,z)\big) \end{aligned}$$

Since the composition of continuous functions is again continuous, H_{02} is a homotopy. Furthermore, $H_{12}(1, H_{01}(1,\cdot))$ is the composition of two biholomorphisms and hence biholomorphic and $H_{02}(0,\cdot) = H_{12}(0, H_{01}(0,\cdot)) = f_2 \circ f_1^{-1}\big(f_1 \circ f_0^{-1}(\cdot)\big) = f_2 \circ f_0^{-1}(\cdot)$. It follows that $f_2 \circ f_0^{-1} : S_0 \to S_2$ is homotopic to a biholomorphism. Hence $(S_0,f_0) \sim (S_2,f_2)$.

The space of all such equivalence classes $\{[S,f] \mid f : R \to S\}$ is identified with $T(R)$.

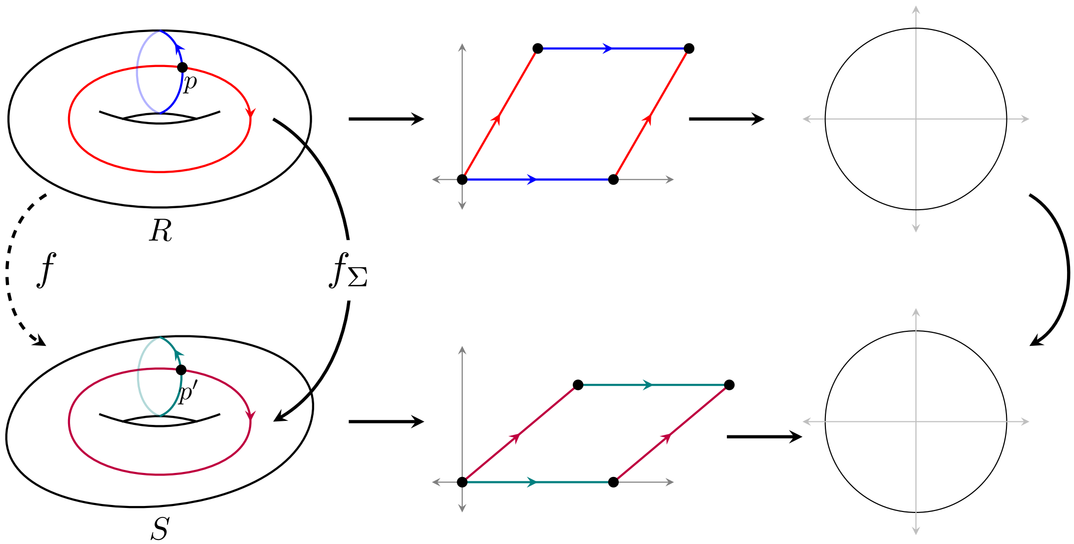

Figure 4: A schematic of the various maps and spaces involved in the proof that $\Phi_\Sigma : T(R) \to T_g$ is surjective. The diffeomorphism between the disks is pulled back to one between the polygons, which is projected to the Riemann surfaces, via the usual quotient map.

2.3 Comparing the Definitions

The above definitions are both based on notions of equivalence which are finer than biholomorphisms. We claim that the resulting spaces can be identified with each other.

In fact, for any marked closed Riemann surface $[R, \Sigma_p] \in T_g$, the map

$$\begin{aligned} \Phi_{\Sigma_p} : T(R) &\longrightarrow T_g \\ [S, f] &\longrightarrow [S, f_*(\Sigma_p)] \end{aligned}$$

is bijective. Until one defined additional structure on the spaces - a metric for example - there is not much we can consider beyond bijective correspondence.

To prove injectivity, one could assume that $[S_0, f_0], [S_1, f_1] \in T(R)$ are such that

$$\Phi_{\Sigma_p}([S_0, f_0]) = [S_0, f_{0*}(\Sigma)] = [S_1, f_{1*}(\Sigma)] = \Phi_{\Sigma_p}([S_1, f_1])$$

We start by constructing a map $g_0 : R \to S_0$ in the same equivalence class as $[S_0, f_0]$. By the definition of equivalent marked Riemann surfaces, there's some biholomorphism $h : S_1 \to S_0$, and some (orientation preserving) homeomorphism $g : S_0 \to S_0$ such that $g_0 := g \circ h \circ f_1$ agrees with f_0 on each $A_j, B_j \in \Sigma$ and is homotopic to the map $h \circ f_1$. Next, by definition, $[S_1, f_1] = [S_0, g_0]$. Now we can compare two maps $R \to S_0$. But,

$$S_0 \backslash \{f_0(A_i), f_0(B_i);\ i = 1, \dots, g\} = S_0 \backslash \{g_0(A_i), g_0(B_i);\ i = 1, \dots, g\} \simeq_h \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\}.$$

Where $\simeq_h$ denotes homeomorphism here. Hence we have two maps f_0, g_0 which agree on the boundary of some region homeomorphic to the unit disk. One could construct a homotopy between the maps on the disk (By using, say, $H(t, z) = t g_0'(z) + (1 - t) f_0'(z)$) the pull back this homotopy to S_0. This procedure confirms that $[S_0, f_0] = [S_0, g_0] = [S_1, f_1]$.

Next, we consider surjectivity. Namely we must show that for any $[S, \Sigma'] \in T_g$, there exists a homeomorphism $f : R \to S$ such that

$$[S, \Sigma'] = \Phi_\Sigma([S, f]) = [S, f_*(\Sigma).]$$

The construction starts with a few assumptions (which can always be met) and a few definitions. Let $\Sigma = \{[A_j], [B_j]\}_{j=1}^g$ and $\Sigma' = \{[A_j'], [B_j']\}_{j=1}^g$, with p_0, p_0' denoting the base points of the fundamental groups associated with Σ and Σ', respectively. Define the sets

$$C = \bigcup_{j=1}^{g} (A_j \cup B_j), \qquad R_0 = R \setminus C$$

$$C' = \bigcup_{j=1}^{g} \left(A'_j \cup B'_j\right), \qquad S_0 = S \setminus C'$$

Note that $C \setminus p$ and $C' \setminus p'$ are diffeomorphic, and furthermore we can construct some diffeomorphism, f_Σ between them which sends each $A_j \setminus p \to A'_j \setminus p'$ and $B_j \to B'_j \setminus p'$ while preserving the orientations. Next, both R_0 and S_0 admit orientation-preserving maps to closed polygons with $2g$ sides. (See Figure 2.) From here, we use that the $2g$-sided polygons corresponding to R_0 and S_0 are both homeomorphic to open disks in the complex plain, $\triangle_R$ and $\triangle_S$, respectively.

The diffeomorphism f_Σ induces a homeomorphism $\partial\overline{\triangle_R} \to \partial\overline{\triangle_S}$. In turn, this is homotopic to some diffeomorphism between the boundaries. This follows from [Hir76, Chapter 8, Theorem 1.9], which we'll call the smoothing theorem. It is always possible to extended a diffeomorphism on the boundary of disks (in $\mathbb{C}$ or $\mathbb{R}^2$) to ones on the interior, hence we have a map $\overline{\triangle_R} \to \overline{\triangle_S}$.

The proof concludes by pulling back the above map to one between the polygons, then projecting that map to one $R \to S$.

These two are not the only popular descriptions of Teichmüller spaces. See [IT92, §1.4] for a description based on quasiconformal mappings and Beltrami coefficients.

3 Connection to Moduli Spaces

Teichmüller spaces are closely related to moduli spaces of Riemann surfaces. In this section, we will describe the correspondence in terms of the mapping class group.

We define a group action on $T(R)$ by $\mathit{Diff}_+(R)$, the set of all homotopy classes $[\varphi]$ of orientation-preserving diffeomorphisms $R \to R$. Elements of $\mathit{Diff}_+(R)$ are called *Teichmüller modular transformations* and each such $[\varphi]$ acts on $T(R)$ by

$$[\varphi] \cdot [S, f] = \left[S, f \circ \varphi^{-1}\right].$$

In the language of T_g, this acts only to change the marking on $[S, f]$. Every closed Riemann surface S of the same genus as R is diffeomorphic to R, so the quotient space

$$T(R)/\mathit{Diff}_+(R)$$

should be identified with $\mathcal{M}_g$, the *moduli space of closed Riemann surfaces of genus g*, i.e. the biholomorphism classes of all Riemann surfaces of genus g. This connection allows one to study the moduli space via the Tiechmüller space, and is one of the primary features of Teichmüller spaces. For example, in [EM11] the Teichmüller space is used to characterize the number of closed geodesics in $\mathcal{M}_g$.

References

[EM11] Alex Eskin and Maryam Mirzakhani. Counting closed geodesics in moduli space. `https://arxiv.org/abs/0811.2362v3`, 2011.

[Hir76] M.W. Hirsh. *Differential Topology.* Springer-Verlang, 1976.

[IT92] Y. Imayoshi and M. Taniguchi. *An introduction to Teichmüller spaces.* Springer-Verlag, Tokyo, 1992. Translated and revised from the Japanese by the authors.