Problem Set 2

Math 205B: Spring Quarter, 2018

1. (a) Suppose that $F : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function with the property that at least one of the partial derivatives $F_z(z, w)$, $F_w(z, w)$ is always nonzero when $F(z, w) = 0$. Define a complex structure on the curve $X = \{(z, w) \in \mathbb{C}^2 : F(z, w) = 0\}$ that makes it into a Riemann surface. Hint. Use the holomorphic form of the implicit function theorem.
(b) Show that the curve $w^2 = \sin z$ is a Riemann surface.

2. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended complex plane and

$$\mathbb{S}^2 = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 + t^2 = 1\}$$

the unit sphere. Define the bijection $\phi : \hat{\mathbb{C}} \to \mathbb{S}^2$ by

$$\phi(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right) \qquad z = x + iy \in \mathbb{C},$$
$$\phi(\infty) = (0, 0, 1).$$

(a) Show that (up to a constant factor) the Euclidean metric on $\mathbb{S}^2$ corresponds under ϕ to the spherical metric on $\hat{\mathbb{C}}$, given by

$$d(z, w) = \frac{|z - w|}{\sqrt{(|z|^2 + 1)(|w|^2 + 1)}},$$

for $z, w \in \mathbb{C}$, with

$$d(z, \infty) = \frac{1}{\sqrt{|z|^2 + 1}}, \qquad d(\infty, \infty) = 0.$$

(b) If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a linear fractional transformation

$$f(z) = \frac{az + b}{cz + d} \qquad ad - bc = 1,$$

define $F : \mathbb{S}^2 \to \mathbb{S}^2$ by $F = \phi \circ f \circ \phi^{-1}$. Show that F is a rotation of $\mathbb{S}^2$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}, \qquad |a|^2 + |c|^2 = 1$$

(c) Deduce that $SO(3) \simeq SU(2)/\{I, -I\}$.