Problem Set 4

Math 205B: Spring Quarter, 2018

1. Let
$$f(z) = \frac{1}{2}\left[\log \Delta(z)\right]', \qquad \Delta(z) = g_2^3(z) - 27g_3^2(z).$$
Show that f is a solution of the Chazy equation
$$f''' - 2ff'' + 3(f')^2 = 0.$$

Hint. Let
$$F_1 = f' - \frac{1}{6}f^2, \qquad F_n = F_{n-1}' - \frac{1}{3}nfF_{n-1} \quad n \geq 2.$$
Show that F_n is a modular form of weight $2n+2$ for $n \geq 1$ and
$$F_3 = f''' - 2ff'' + 3(f')^2 - 4\left(f' - \frac{1}{6}f^2\right)^2.$$
Deduce that the left-hand side of the Chazy equation is a modular form of weight 8, determine its limit as $z \to i\infty$, and use the fact that g_2, g_3 have weight 4, 6 and generate the ring of modular forms.

2. Verify the following table and fundamental domain of $\Gamma(2)$ discussed in class (taken from *Elliptic Functions*, K. Chandrasekharan, Springer-Verlag, 1985). You don't have to write anything, just verifty that it all works.

a	b	c	d	$\tau^* = \frac{a\tau+b}{c\tau+d}$	$e_1^* = \wp\left(\frac{\omega_1^*}{2}\right)$	$e_2^* = \wp\left(\frac{\omega_2^*}{2}\right)$	$e_3^* = \wp\left(\frac{\omega_1^*+\omega_2^*}{2}\right)$	$\lambda(\tau^*) = \frac{e_3^*-e_2^*}{e_1^*-e_2^*}$
1	0	0	1	τ	e_1	e_2	e_3	$\lambda(\tau)$
1	1	0	1	$\tau+1$	e_1	e_3	e_2	$\frac{\lambda(\tau)}{\lambda(\tau)-1}$
0	-1	1	0	$-\frac{1}{\tau}$	e_2	e_1	e_3	$1-\lambda(\tau)$
1	-1	1	0	$\frac{\tau-1}{\tau}$	e_2	e_3	e_1	$\frac{\lambda(\tau)-1}{\lambda(\tau)}$
1	0	1	1	$\frac{\tau}{\tau+1}$	e_3	e_2	e_1	$\frac{1}{\lambda(\tau)}$
0	1	-1	1	$\frac{1}{1-\tau}$	e_3	e_1	e_2	$\frac{1}{1-\lambda(\tau)}$

It follows that *if a, b, c, d are integers such that $ad-bc=1$, then*

$$\lambda\left(\frac{a\tau+b}{c\tau+d}\right)=\lambda(\tau), \qquad \operatorname{Im}\tau>0,$$

if and only if b and c are even, so that a and d are odd.

All transformations

$$\tau\to\tau^*=\frac{a\tau+b}{c\tau+d}, \qquad \operatorname{Im}\tau>0, \quad ad-bc=1,$$

where b and c are even, while a and d are odd, form a group, which we shall denote by G. *Then G is a sub-group of the modular group Γ, of finite index.* By (7.7), the function $\lambda(\tau)$ is invariant under all transformations belonging to G. *Hence $\lambda(\tau)$ is a modular function.*

If A denotes the transformation $\tau\to\tau+1$, and B denotes the transformation $\tau\to-\frac{1}{\tau}$, then the transformations: $A^2:\tau\to\tau+2$, and $(BA^2B):\tau\to\frac{\tau}{1-2\tau}$ belong to G. In fact we have

Lemma 1. *The group G is generated by the transformations*

$$\tau\to\tau+2, \qquad \tau\to\frac{\tau}{1-2\tau}. \tag{7.8}$$

Proof. Let $M\in G$, where

$$M\tau=\frac{a\tau+b}{c\tau+d}, \quad ad-bc=1, \quad \left.\begin{matrix} b\equiv c\equiv 0 \pmod 2 \\ a\equiv d\equiv 1 \pmod 2\end{matrix}\right\}.$$

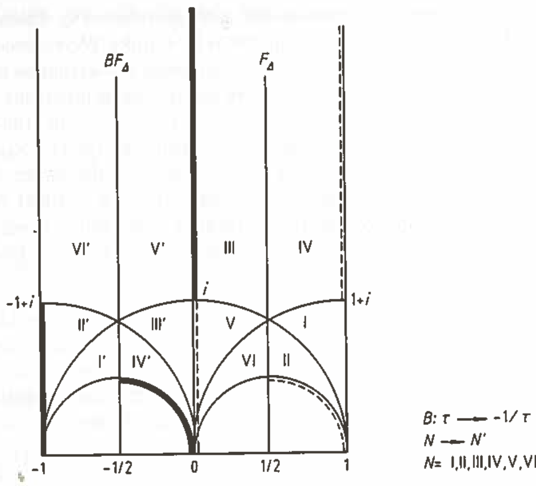

Fig. 9

Notes on Chapter VII

§§ 1–3. There are many different ways of defining the Jacobian elliptic functions. As mentioned already in the Notes on Chapters I and III, they were originally discovered by the inversion of integrals of the type

$$u = \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad k^2 \neq 0, 1,$$

(cf. Remarks, § 4), some thirty years before the Weierstrassian elliptic functions. One can define them, after Hurwitz, by means of the sigma-functions as in (3.2). Defining them as quotients of theta-functions makes for simpler proofs.

The function $k^2(\tau)$ – the square of the modulus – was studied by Abel, Jacobi, and Hermite. See the references to Hermite in Dedekind's paper cited in the Notes on Chapter VI. See also § 54 of H.A. Schwarz's book (cited in the Notes on Ch. III).

§ 4. The problem of solving the equation $k^2(\tau) = a$ in (4.8) for a given a, $a \neq 0, 1$, is referred to as the *problem of inversion* for k^2. The solution here effected, by an appeal to Theorem 6 of Chapter VI, is due to Hurwitz. Jacobi's original proof (*Ges. Werke*, I, p. 520) holds only for $0 < a < 1$, as was noted by Weierstrass (ibid. p. 545), who filled the gap, *Werke*, II, 257–309.

§ 5. The infinite products for sn u, cn u, dn u, are here derived from the infinite products for theta-functions obtained in Chapter V. This reverses, of course, the historical development.