**Problem Set 5**

Math 205B: Spring Quarter, 2018

**1.** Let $p : \mathcal{O} \to \mathbb{C}$ be the map $p : f_z \in \mathcal{O}_z \mapsto z$. Show that $p$ is open and continuous. If

$$N(f, U) = \{f_z \in \mathcal{O}_z : z \in U,\ (f, U) \in f_z\}$$

is an open set in $\mathcal{O}$ defined by the function element $(f, U)$, conclude that

$$p|_{N(f,U)} : N(f, U) \to U$$

is a homeomorphism, meaning that the Hausdorff topological space $\mathcal{O}$ is a two-dimensional (disconnected, not first-countable) manifold.

**2.** Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve and $P : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ a polynomial. Suppose that $f_a, g_a \in \mathcal{O}_a$ are germs at $\gamma(0) = a$ that can be continued analytically along $\gamma$ to give germs $f_{\gamma(t)}, g_{\gamma(t)} \in \mathcal{O}_{\gamma(t)}$ for $0 \leq t \leq 1$. If $P(f_a, g_a) = 0$, show that

$$P\left(f_{\gamma(t)}, g_{\gamma(t)}\right) = 0 \qquad \text{for all } 0 \leq t \leq 1.$$

(This result is an example of *permanence of relations* under analytic continuation.)

**3.** Suppose that $X$ is a Riemann surface and the continuous map $p : X' \to X$ is a covering of $X$. Show that there is a complex structure on $X'$ such that $p : X' \to X$ is holomorphic.

**4.** (a) Define $d : \mathcal{O} \to \mathcal{O}$ by $f_a \mapsto df_a$ where $df_a = [(f', U)]$ if $f_a = [(f, U)]$. Show that $d$ is a covering map.

(b) Let $\Omega \subset \mathbb{C}$ be a simply connected open set and $f : \Omega \to \mathbb{C}$ a holomorphic function. Show that $f$ has a primitive $F : \Omega \to \mathbb{C}$ such that $F' = f$.

Hint. Lift $f$ along curves in $\Omega$ with respect to $d$ and use the monodromy theorem.