CHAPTER 1

Measures

We begin with some preliminary definitions and terminology related to measures on arbitrary sets.

1.1. Sets

We use standard definitions and notations from set theory and will assume the axiom of choice when needed. The words 'collection' and 'family' are synonymous with 'set' — we use them when talking about sets of sets. We denote the collection of subsets, or power set, of a set X by $\mathcal{P}(X)$. The notation 2^X is also used.

If $E \subset X$ and the set X is understood, we denote the complement of E in X by $E^c = X \setminus E$. De Morgan's laws state that

$$\left(\bigcup_{\alpha \in I} E_\alpha\right)^c = \bigcap_{\alpha \in I} E_\alpha^c, \qquad \left(\bigcap_{\alpha \in I} E_i\right)^c = \bigcup_{\alpha \in I}^{\infty} E_\alpha^c.$$

We say that a collection

$$\mathcal{C} = \{E_\alpha \subset X : \alpha \in I\}$$

of subsets of a set X, indexed by a set I, covers $E \subset X$ if

$$\bigcup_{\alpha \in I} E_\alpha \supset E.$$

The collection $\mathcal{C}$ is disjoint if $E_\alpha \cap E_\beta = \varnothing$ for $\alpha \neq \beta$.

1.2. Extended real numbers

It is convenient to use the extended real numbers

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

This allows us, for example, to talk about sets with infinite measure or non-negative functions with infinite integral. The extended real numbers are totally ordered in the obvious way: ∞ is the largest element, $-\infty$ is the smallest element, and real numbers are ordered as in $\mathbb{R}$. Algebraic operations on $\overline{\mathbb{R}}$ are defined when they are unambiguous *e.g.* $\infty + x = \infty$ for every $x \in \overline{\mathbb{R}}$ except $x = -\infty$, but $\infty - \infty$ is undefined.

We define a topology on $\overline{\mathbb{R}}$ in a natural way, making $\overline{\mathbb{R}}$ homeomorphic to a compact interval. For example, the function $\phi : \overline{\mathbb{R}} \to [-1, 1]$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = \infty \\ x/\sqrt{1+x^2} & \text{if } -\infty < x < \infty \\ -1 & \text{if } x = -\infty \end{cases}$$

is a homeomorphism.

A primary reason to use the extended real numbers is that upper and lower bounds always exist. Every non-empty subset of $\overline{\mathbb{R}}$ has a supremum (equal to ∞ if the subset contains ∞ or is not bounded from above in $\mathbb{R}$) and infimum (equal to $-\infty$ if the subset contains $-\infty$ or is not bounded from below in $\mathbb{R}$). Every increasing sequence of extended real numbers converges to its supremum, and every decreasing sequence converges to its infimum. Every sum $\sum_{i=1}^{\infty} x_i$ with non-negative terms $x_i \geq 0$ converges in $\overline{\mathbb{R}}$ (to ∞ if $x_i = \infty$ for some $i \in \mathbb{N}$ or the series diverges in $\mathbb{R}$):

$$\sum_{i=1}^{\infty} x_i = \sup \left\{ \sum_{i \in F} x_i : F \subset \mathbb{N} \text{ is finite} \right\}.$$

As for non-negative sums of real numbers, non-negative sums of extended real numbers are unconditionally convergent (the order of the terms does not matter); we can rearrange sums of non-negative extended real numbers

$$\sum_{i=1}^{\infty} (x_i + y_i) = \sum_{i=1}^{\infty} x_i + \sum_{i=1}^{\infty} y_i;$$

and double sums may be evaluated as iterated single sums

$$\begin{aligned} \sum_{i,j=1}^{\infty} x_{ij} &= \sup \left\{ \sum_{(i,j) \in F} x_{ij} : F \subset \mathbb{N} \times \mathbb{N} \text{ is finite} \right\} \\ &= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} x_{ij} \right) \\ &= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} x_{ij} \right). \end{aligned}$$

The use of extended real numbers is closely tied to the order and monotonicity properties of $\mathbb{R}$. In dealing with complex numbers or elements of a vector space, we will always require that they are strictly finite.

1.3. Outer measures

As stated in the following definition, an outer measure is a monotone, countably subadditive, non-negative, extended real-valued function defined on all subsets of a set.

Definition 1.1. An outer measure μ^* on a set X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that:

(a) $\mu^*(\varnothing) = 0$;
(b) if $E \subset F \subset X$, then $\mu^*(E) \leq \mu^*(F)$;
(c) if $\{E_i \subset X : i \in \mathbb{N}\}$ is a countable collection of subsets of X, then

$$\mu^* \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

We obtain a statement about finite unions from a statement about infinite unions by taking all but finitely many sets in the union equal to the empty set.

1.4. σ-algebras

A σ-algebra is a collection of subsets of a set that is closed under the operations of taking complements, countable unions, and countable intersections.

Definition 1.2. A σ-algebra on a set X is a collection $\mathcal{A}$ of subsets of X such that:

(a) $\varnothing, X \in \mathcal{A}$;
(b) if $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$;
(c) if $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$ then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}, \qquad \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}.$$

From de Morgan's laws, a collection of subsets is σ-algebra if it contains $\varnothing$ and is closed under the operations of taking complements and countable unions (or, equivalently, countable intersections).

Example 1.3. If X is a set, then $\{\varnothing, X\}$ and $\mathcal{P}(X)$ are σ-algebras on X; they are the smallest and largest σ-algebras on X, respectively.

If $\mathcal{F}$ is any collection of subsets of a set X, then there is a smallest σ-algebra on X that contains $\mathcal{F}$, denoted by $\sigma(\mathcal{F})$, which is obtained by taking the intersection of all σ-algebras that contain $\mathcal{F}$. (This intersection is nonempty, since $\mathcal{P}(X)$ is a σ-algebra that contains $\mathcal{F}$, and an intersection of σ-algebras is a σ-algebra.)

1.5. Measures

Measurable spaces provide the domain of measures.

Definition 1.4. A measurable space $(X, \mathcal{A})$ is a non-empty set X equipped with a σ-algebra $\mathcal{A}$ on X.

A measure is a countably additive, non-negative, extended real-valued function defined on a σ-algebra.

Definition 1.5. A measure μ on a measurable space $(X, \mathcal{A})$ is a function

$$\mu : \mathcal{A} \to [0, \infty]$$

such that

(a) $\mu(\varnothing) = 0$;
(b) if $\{A_i \in \mathcal{A} : i \in \mathbb{N}\}$ is a countable disjoint collection of sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

A measure space $(X, \mathcal{A}, \mu)$ consists of a set X, a σ-algebra $\mathcal{A}$ on X, and a measure μ defined on $\mathcal{A}$. When $\mathcal{A}$ and μ are clear from the context, we will refer to the measure space X.

A useful implication of countable additivity is the following monotonicity result.

Proposition 1.6. *If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence of measurable sets, meaning that $A_{i+1} \supset A_i$, then*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i).$$

If $\{A_i : i \in \mathbb{N}\}$ *is a decreasing sequence of measurable sets, meaning that* $A_{i+1} \subset A_i$*, and* $\mu(A_1) < \infty$*, then*

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

A set of measure zero, or a null set, is a measurable set N such that $\mu(N) = 0$. A property which holds for all $x \in X \setminus M$ where M is a subset of a set of measure zero is said to hold almost everywhere, or a.e. for short. In general, a subset of a set of measure zero need not be measurable, but if it is, it must have measure zero. It is frequently convenient to use measure spaces which are complete in the following sense.[1]

Definition 1.7. A measure space $(X, \mathcal{A}, \mu)$ is complete if every subset of a set of measure zero is also measurable.

Note that completeness depends on the measure μ, not just the σ-algebra $\mathcal{A}$. Any measure space $(X, \mathcal{A}, \mu)$ is contained in a uniquely defined completion $(X, \overline{\mathcal{A}}, \overline{\mu})$, which the smallest complete measure space that contains it and is given explicitly as follows.

Theorem 1.8. *If* $(X, \mathcal{A}, \mu)$ *is a measure space, define* $(X, \overline{\mathcal{A}}, \overline{\mu})$ *by*

$$\overline{\mathcal{A}} = \{A \cup B : A \in \mathcal{A},\ B \subset N \text{ where } N \in \mathcal{A} \text{ satisfies } \mu(N) = 0\}$$

with $\overline{\mu}(A \cup B) = \mu(A)$*. Then* $(X, \overline{\mathcal{A}}, \overline{\mu})$ *is a complete measure space such that* $\overline{\mathcal{A}} \supset \mathcal{A}$ *and* $\overline{\mu}$ *is the unique extension of* μ *to* $\overline{\mathcal{A}}$*.*

See Theorem 1.9 in [**4**] for the proof.

[1] This is, of course, a different sense of 'complete' than the one used in talking about complete metric spaces.