1. Suppose that $\mu$ is a measure on a set $X$. If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence of measurable sets ($A_{i+1} \supset A_i$), show that

$$\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \lim_{i \to \infty} \mu(A_i).$$

If $\{A_i : i \in \mathbb{N}\}$ is a decreasing sequence of measurable sets ($A_{i+1} \subset A_i$) and $\mu(A_1) < \infty$, show that

$$\mu \left( \bigcap_{i=1}^{\infty} A_i \right) = \lim_{i \to \infty} \mu(A_i).$$

Give an example to show that the last result need not hold unless we assume that some set $A_i$ has finite measure.

2. Suppose that $A \subset \mathbb{R}^n$ satisfies

$$\mu^* (R) = \mu^* (R \cap A) + \mu^* (R \setminus A)$$

for every rectangle $R$, where $\mu^*$ denotes Lebesgue outer measure. Prove that $A$ is Lebesgue measurable.

3. (a) Show that the standard Cantor set has Lebesgue measure zero.
(b) Give an example of a subset of $[0, 1]$ that has non-zero Lebesgue measure but does not contain any nonempty open intervals.
(c) Show that the $x$-axis

$$E = \{(x, 0) : x \in \mathbb{R}\}$$

has zero (two-dimensional) Lebesgue measure.

4. Define the distance between sets $A, B \subset \mathbb{R}^n$ by

$$d(A, B) = \inf \{|x - y| : x \in A, y \in B\}$$

where $|\cdot|$ denotes the Euclidean norm. If $A, B$ are not necessarily measurable subsets of $\mathbb{R}^n$ and $d(A, B) > 0$, prove that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

where $\mu^*$ denotes Lebesgue outer measure.