1. If \( f_1 \geq f_2 \geq f_3 \geq \ldots \) is a monotone decreasing sequence of extended real-valued functions on a measure space and \( \int f_1 \, d\mu < \infty \), show that
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu.
\]
Give an example to show this result need not be true if we omit the assumption that \( f_1 \) is integrable.

2. Prove the following generalization of the dominated convergence theorem:
Let \( \{f_n\} \) be a sequence of measurable, real-valued functions such that \( f_n \to f \) pointwise as \( n \to \infty \). If \( |f_n| \leq g_n \), where \( g_n \) is integrable and
\[
\lim_{n \to \infty} \int g_n \, d\mu = \int g \, d\mu
\]
for some integrable function \( g \), then
\[
\lim_{n \to \infty} \int |f_n - f| \, d\mu = 0.
\]

3. If \( f_n, f \) are integrable functions such that \( f_n \to f \) uniformly on a finite measure space \( X \), prove that \( \int f_n \, d\mu \to \int f \, d\mu \). Use Egoroff’s theorem (Problem 4 in Set 3) and the absolute continuity of the integral proved in class to give an alternative proof of the dominated convergence theorem.

4. Let \( \{f_n\} \) be a sequence of measurable, real-valued functions on a measure space \( X \) such that \( f_n \to f \) pointwise, where \( f : X \to \mathbb{R} \), and suppose that for some constant \( M > 0 \)
\[
\int |f_n| \, d\mu \leq M \quad \text{for all } n \in \mathbb{N}.
\]
(a) Show that
\[
\int |f| \, d\mu \leq M.
\]
Give an example to show that we may have \( \int |f_n| \, d\mu = M \) for every \( n \in \mathbb{N} \) but \( \int |f| \, d\mu < M \).
(b) Show that
\[
\lim_{n \to \infty} \int ||f_n| - |f| - |f_n - f|| \, d\mu = 0.
\]
HINT. Show that \( ||a + b| - |b|| \leq |a| \). (This quantifies the ‘loss’ of mass in the integral when \( n \to \infty \).)
5. Let \((X, A, \mu)\) be a complete measure space with finite measure, \(\mu(X) < \infty\). Denote by \(M(X)\) the space of (equivalence classes of) measurable functions \(f : X \to \mathbb{R}\), where we identify functions that are equal \(\mu\)-a.e.. For \(f, g \in M(X)\), define
\[
d(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} \, d\mu.
\]

(a) Show that \(d\) is a metric on \(M(X)\).

(b) A sequence \(f_n \to f\) converges in measure if for every \(\epsilon > 0\)
\[
\lim_{n \to \infty} \mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = 0.
\]

Show that \(f_n \to f\) in measure if and only if \(d(f_n, f) \to 0\) as \(n \to \infty\).