Introduction to Dynamical Systems

John K. Hunter

Department of Mathematics, University of California at Davis

Contents

CHAPTER 1

Introduction

We will begin by discussing some general properties of initial value problems (IVPs) for ordinary differential equations (ODEs) as well as the basic underlying mathematical theory.

1.1. First-order systems of ODEs

> Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?
>
> Edward Lorenz, 1972

We consider an autonomous system of first-order ODEs of the form

$$x_t = f(x) \tag{1.1}$$

where $x(t) \in \mathbb{R}^d$ is a vector of dependent variables, $f : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field, and x_t is the time-derivative, which we also write as dx/dt or $\dot{x}$. In component form, $x = (x_1, \dots, x_d)$,

$$f(x) = (f_1(x_1 \dots, x_d), \dots, f_n(x_1, \dots x_d)),$$

and the system is

$$\begin{aligned} x_{1t} &= f_1(x_1 \dots, x_d), \\ x_{2t} &= f_2(x_1 \dots, x_d), \\ &\dots, \\ x_{dt} &= f_d(x_1 \dots, x_d). \end{aligned}$$

We may regard (1.1) as describing the evolution in continuous time t of a dynamical system with finite-dimensional state $x(t)$ of dimension d.

Autonomous ODEs arise as models of systems whose laws do not change in time. They are invariant under translations in time: if $x(t)$ is a solution, then so is $x(t + t_0)$ for any constant t_0.

Example 1.1. The Lorenz system for $(x, y, z) \in \mathbb{R}^3$ is

$$\begin{aligned} x_t &= \sigma(y - x), \\ y_t &= rx - y - xz, \\ z_t &= xy - \beta z. \end{aligned} \tag{1.2}$$

The system depends on three positive parameters σ, r, β; a commonly studied case is $\sigma = 10$, $r = 28$, and $\beta = 4/3$. Lorenz (1963) obtained (1.2) as a truncated model of thermal convection in a fluid layer, where σ has the interpretation of a Prandtl number(the ratio of kinematic viscosity and thermal diffusivity), r corresponds to a Rayleigh number, which is a dimensionless parameter proportional to the temperature difference across the fluid layer and the gravitational acceleration acting on the fluid, and β is a ratio of the height and width of the fluid layer.

Lorenz discovered that solutions of (1.2) may behave chaotically, showing that even low-dimensional nonlinear dynamical systems can behave in complex ways. Solutions of chaotic systems are sensitive to small changes in the initial conditions, and Lorenz used this model to discuss the unpredictability of weather (the "butterfly effect").

If $\bar{x} \in \mathbb{R}^d$ is a zero of f, meaning that

$$f(\bar{x}) = 0, \tag{1.3}$$

then (1.1) has the constant solution $x(t) = \bar{x}$. We call $\bar{x}$ an *equilibrium solution*, or *steady state solution*, or *fixed point* of (1.1). An equilibrium may be stable or unstable, depending on whether small perturbations of the equilibrium decay — or, at least, remain bounded — or grow. (See Definition 1.14 below for a precise definition.) The determination of the stability of equilibria will be an important topic in the following.

Other types of ODEs can be put in the form (1.1). This rewriting does not simplify their analysis, and may obscure the specific structure of the ODEs, but it shows that (1.1) is rather a general form.

EXAMPLE 1.2. A non-autonomous system for $x(t) \in \mathbb{R}^d$ has the form

$$x_t = f(x, t) \tag{1.4}$$

where $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. A nonautonomous ODE describes systems governed by laws that vary in time *e.g.* due to external influences. Equation (1.4) can be written as an autonomous ('suspended') system for $y = (x, s) \in \mathbb{R}^{n+1}$ with $s = t$ as

$$x_t = f(x, s), \qquad s_t = 1.$$

Note that this increases the order of the system by one, and even if the original system has an equilibrium solution $x(t) = \bar{x}$ such that $f(\bar{x}, t) = 0$, the suspended system has no equilibrium solutions for y.

Higher-order ODEs can be written as first order systems by the introduction of derivatives as new dependent variables.

EXAMPLE 1.3. A second-order system for $x(t) \in \mathbb{R}^d$ of the form

$$x_{tt} = f(x, x_t) \tag{1.5}$$

can be written as a first-order system for $z = (x, y) \in \mathbb{R}^{2d}$ with $y = x_t$ as

$$x_t = y, \qquad y_t = f(x, y).$$

Note that this doubles the dimension of the system.

EXAMPLE 1.4. In Newtonian mechanics, the position $x(t) \in \mathbb{R}^d$ of a particle of mass m moving in d space dimensions in a spatially-dependent force-field $F(x)$, such as a planet in motion around the sun, satisfies

$$mx_{tt} = F(x).$$

If $p = mx_t$ is the momentum of the particle, then (x, p) satisfies the first-order system

$$x_t = \frac{1}{m}p, \qquad p_t = F(x). \tag{1.6}$$

A conservative force-field is derived from a potential $V : \mathbb{R}^d \to \mathbb{R}$,

$$F = -\frac{\partial V}{\partial x}, \qquad (F_1, \ldots, F_d) = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d}\right).$$

We use $\partial/\partial x$, $\partial/\partial p$ to denote the derivatives, or gradients with respect to x, p respectively. In that case, (1.6) becomes the Hamiltonian system

$$x_t = \frac{\partial H}{\partial p}, \qquad p_t = -\frac{\partial H}{\partial x} \tag{1.7}$$

where the Hamiltonian

$$H(x, p) = \frac{1}{2m}p^2 + V(x)$$

is the total energy (kinetic + potential) of the particle. The Hamiltonian is a conserved quantity of (1.7), since by the chain rule

$$\begin{aligned} \frac{d}{dt}H\left(x(t), p(t)\right) &= \frac{\partial H}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial H}{\partial p} \cdot \frac{dx}{dt} \\ &= -\frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \frac{\partial H}{\partial x} \\ &= 0. \end{aligned}$$

Thus, solutions (x, p) of (1.7) lie on the level surfaces $H(x, p) = \text{constant}$.

1.2. Existence and uniqueness theorem for IVPs

An initial value problem (IVP) for (1.1) consists of solving the ODE subject to an initial condition (IC) for x:

$$\begin{aligned} x_t &= f(x), \\ x(0) &= x_0. \end{aligned} \tag{1.8}$$

Here, $x_0 \in \mathbb{R}^d$ is a given constant vector. For an autonomous system, there is no loss of generality in imposing the initial condition at $t = 0$, rather than some other time $t = t_0$.

For a first-order system, we impose initial data for x. For a second-order system, such as (1.5), we impose initial data for x and x_t, and analogously for higher-order systems. The ODE in (1.8) determines $x_t(0)$ from x_0, and we can obtain all higher order derivatives $x^{(n)}(0)$ by differentiating the equation with respect to t and evaluating the result at $t = 0$. Thus, it reasonable to expect that (1.8) determines a unique solution, and this is indeed true provided that $f(x)$ satisfies a mild smoothness condition, called Lipschitz continuity, which is nearly always met in applications. Before stating the existence-uniqueness theorem, we explain what Lipschitz continuity means.

We denote by

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

the Euclidean norm of a vector $x \in \mathbb{R}^d$.

Definition 1.5. A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is *locally Lipschitz continuous* on $\mathbb{R}^d$, or Lipschitz continuous for short, if for every $R > 0$ there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \qquad \text{for all } x, y \in \mathbb{R}^d \text{ such that } |x|, |y| \leq R.$$

We refer to M as a Lipschitz constant for f.

A sufficient condition for $f = (f_1, \dots, f_d)$ to be a locally Lipschitz continuous function of $x = (x_1, \dots, x_d)$ is that f is continuous differentiable (C^1), meaning that all its partial derivatives

$$\frac{\partial f_i}{\partial x_j}, \qquad 1 \leq i, j \leq d$$

exist and are continuous functions.

To show this, note that from the fundamental theorem of calculus

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \frac{d}{ds} f\left(y + s(x - y)\right) \, ds \\ &= \int_0^1 Df\left(y + s(x - y)\right)(x - y) \, ds. \end{aligned}$$

Here Df is the derivative of f, whose matrix is the Jacobian matrix of f with components $\partial f_i / \partial x_j$. Hence

$$\begin{aligned} |f(x) - f(y)| &\leq \int_0^1 \left|Df\left(y + s(x - y)\right)(x - y)\right| \, ds \\ &\leq \left(\int_0^1 \left\|Df\left(y + s(x - y)\right)\right\| \, ds\right) |x - y| \\ &\leq M|x - y| \end{aligned}$$

where $\|Df\|$ denotes the Euclidean matrix norm of Df and

$$M = \max_{0 \leq s \leq 1} \left\|Df\left(y + s(x - y)\right)\right\|.$$

For scalar-valued functions, this result also follows from the mean value theorem.

Example 1.6. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is locally Lipschitz continuous on $\mathbb{R}$, since it is continuously differentiable. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = |x|$ is Lipschitz continuous, although it is not differentiable at $x = 0$. The function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = |x|^{1/2}$ is not Lipschitz continuous at $x = 0$, although it is continuous.

The following result, due to Picard and Lindelöf, is the fundamental local existence and uniqueness theorem for IVPs for ODEs. It is a local existence theorem because it only asserts the existence of a solution for sufficiently small times, not necessarily for all times.

Theorem 1.7 (Existence-uniqueness). *If $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous, then there exists a unique solution $x : I \to \mathbb{R}^d$ of (1.8) defined on some time-interval $I \subset \mathbb{R}$ containing $t = 0$.*

In practice, to apply this theorem to (1.8), we usually just have to check that the right-hand side $f(x)$ is a continuously differentiable function of the dependent variables x.

We will not prove Theorem 1.7 here, but we explain the main idea of the proof. Since it is impossible, in general, to find an explicit solution of a nonlinear IVP such as (1.8), we have to construct the solution by some kind of approximation procedure. Using the method of Picard iteration, we rewrite (1.8) as an equivalent integral equation

$$x(t) = x_0 + \int_0^t f\left(x(s)\right)\, ds. \tag{1.9}$$

This integral equation formulation includes both the initial condition and the ODE. We then define a sequence $x_n(t)$ of functions by iteration, starting from the constant initial data x_0:

$$x_{n+1}(t) = x_0 + \int_0^t f\left(x_n(s)\right)\, ds, \qquad n = 1, 2, 3, \dots. \tag{1.10}$$

Using the Lipschitz continuity of f, one can show that this sequence converges uniformly on a sufficiently small time interval I to a unique function $x(t)$. Taking the limit of (1.10) as $n \to \infty$, we find that $x(t)$ satisfies (1.9), so it is the solution of (1.8).

Two simple scalar examples illustrate Theorem 1.7. The first example shows that solutions of nonlinear IVPs need not exist for all times.

Example 1.8. Consider the IVP

$$x_t = x^2, \qquad x(0) = x_0.$$

For $x_0 \neq 0$, we find by separating variables that the solution is

$$x(t) = -\left(\frac{1}{t - 1/x_0}\right). \tag{1.11}$$

If $x_0 > 0$, the solution exists only for $-\infty < t < t_0$ where $t_0 = 1/x_0$, and $x(t) \to -\infty$ as $t \to t_0$. Note that the larger the initial data x_0 the smaller the 'blow-up' time t_0. If $x_0 < 0$, then $t_0 < 0$ and the solution exists for $t_0 < t < \infty$. Only if $x_0 = 0$ does the solution $x(t) = 0$ exists for all times $t \in \mathbb{R}$.

One might consider using (1.11) past the time t_0, but continuing a solution through infinity does not make much sense in evolution problems. In applications, the appearance of a singularity typically signifies that the assumptions of the mathematical model have broken down in some way.

The second example shows that solutions of (1.8) need not be unique if f is not Lipschitz continuous.

Example 1.9. Consider the IVP

$$x_t = |x|^{1/2}, \qquad x(0) = 0. \tag{1.12}$$

The right-hand side of the ODE, $f(x) = |x|^{1/2}$, is not differentiable or Lipschitz continuous at the initial data $x = 0$. One solution is $x(t) = 0$, but this is not the only solution. Separating variables in the ODE, we get the solution

$$x(t) = \frac{1}{4}(t - t_0)^2.$$

Thus, for any $t_0 \geq 0$, the function

$$x(t) = \begin{cases} 0 & \text{if } t \leq t_0 \\ (1/4)(t - t_0)^2 & \text{if } t > t_0 \end{cases}$$

is also a solution of the IVP (1.12). The parabolic solution can 'take off' spontanteously with zero derivative from the zero solution at any nonnegative time t_0. In applications, a lack of uniqueness typically means that something is missing from the mathematical model.

If $f(x)$ is only assumed to be a continuous function of x, then solutions of (1.8) always exist (this is the Peano existence theorem) although they may fail to be unique, as shown by Example 1.9. In future, we will assume that f is a smooth function; typically, f will be C^∞, meaning that is has continuous derivatives of all orders. In that case, the issue of non-uniqueness does not arise.

Even for arbitrarily smooth functions f, the solution of the nonlinear IVP (1.8) may fail to exist for all times if $f(x)$ grows faster than a linear function of x, as in Example 1.8. According to the following theorem, the only way in which global existence can fail is if the solution 'escapes' to infinity. We refer to this phenomenon informally as 'blow-up.'

Theorem 1.10 (Extension). *If $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous, then the solution $x : I \to \mathbb{R}^d$ of (1.8) exists on a maximal time-interval*

$$I = (T_-, T_+) \subset \mathbb{R}$$

where $-\infty \leq T_- < 0$ and $0 < T_+ \leq \infty$. If $T_+ < \infty$, then $|x(t)| \to \infty$ as $t \uparrow T_+$, and if $T_- > -\infty$, then $|x(t)| \to \infty$ as $t \downarrow T_-$,

This theorem implies that we can continue a solution of the ODE so long as it remains bounded.

Example 1.11. Consider the function defined for $t \neq 0$ by

$$x(t) = \sin\left(\frac{1}{t}\right).$$

This function cannot be extended to a differentiable, or even continuous, function at $t = 0$ even though it is bounded. This kind of behavior cannot happen for solutions of ODEs with continuous right-hand sides, because the ODE implies that the derivative x_t remains bounded if the solution x remains bounded. On the other hand, an ODE may have a solution like $x(t) = 1/t$, since the derivative x_t only becomes large when x itself becomes large.

Example 1.12. Theorem 1.7 implies that the Lorenz system (1.1) with arbitrary initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0$$

has a unique solution defined on some time interval containing 0, since the right hand side is a smooth (in fact, quadratic) function of (x, y, z). The theorem does not imply, however, that the solution exists for all t.

Nevertheless, we claim that when the parameters (σ, r, β) are positive the solution exists for all $t \geq 0$. From Theorem 1.10, this conclusion follows if we can show that the solution remains bounded, and to do this we introduce a suitable Lyapunov function. A convenient choice is

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2.$$

Using the chain rule, we find that if (x, y, z) satisfies (1.2), then

$$\begin{aligned}\frac{d}{dt}V(x,y,z) &= 2rxx_t + 2\sigma yy_t + 2\sigma(z-2r)z_t \\ &= 2r\sigma x(y-x) + 2\sigma y(rx - y - xz) + 2\sigma(z-2r)(xy - \beta z) \\ &= -2\sigma\left[rx^2 + y^2 + \beta(z-r)^2\right] + 2\beta\sigma r^2.\end{aligned}$$

Hence, if $W(x, y, z) > \beta r^2$, where

$$W(x,y,z) = rx^2 + y^2 + \beta(z-r)^2,$$

then $V(x, y, z)$ is decreasing in time. This means that if C is sufficiently large that the ellipsoid $V(x, y, z) < C$ contains the ellipsoid $W(x, y, z) \le \beta r^2$, then solutions cannot escape from the region $V(x, y, z) < C$ forward in time, since they move 'inwards' across the boundary $V(x, y, z) = C$. Therefore, the solution remains bounded and exists for all $t \ge 0$.

Note that this argument does not preclude the possibility that solutions of (1.2) blow up backwards in time. The Lorenz system models a forced, dissipative system and its dynamics are not time-reversible. (This contrasts with the dynamics of conservative, Hamiltonian systems, which are time-reversible.)

1.3. Linear systems of ODEs

An IVP for a (homogeneous, autonomous, first-order) linear system of ODEs for $x(t) \in \mathbb{R}^d$ has the form

$$\begin{aligned} x_t &= Ax, \\ x(0) &= x_0 \end{aligned} \tag{1.13}$$

where A is a $d \times d$ matrix and $x_0 \in \mathbb{R}^d$. This system corresponds to (1.8) with $f(x) = Ax$. Linear systems are much simpler to study than nonlinear systems, and perhaps the first question to ask of any equation is whether it is linear or nonlinear.

The linear IVP (1.13) has a unique global solution, which is given explicitly by

$$x(t) = e^{tA}x_0, \qquad -\infty < t < \infty$$

where

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \dots + \frac{1}{n!}t^nA^n + \dots$$

is the matrix exponential.

If A is nonsingular, then (1.13) has a unique equilibrium solution $x = 0$. This equilibrium is stable if all eigenvalues of A have negative real parts and unstable if some eigenvalue of A has positive real part. If A is singular, then there is a ν-dimensional subspace of equilibria where ν is the nullity of A.

Linear systems are important in their own right, but they also arise as approximations of nonlinear systems. Suppose that $\bar{x}$ is an equilibrium solution of (1.1), satisfying (1.3). Then writing

$$x(t) = \bar{x} + y(t)$$

and Taylor expanding $f(x)$ about $\bar{x}$, we get

$$f(\bar{x} + y) = Ay + \dots$$

where A is the derivative of f evaluated at $\bar{x}$, with matrix (a_{ij}):

$$A = Df(\bar{x}), \qquad a_{ij} = \frac{\partial f_i}{\partial x_j}(\bar{x}).$$

The linearized approximation of (1.1) at the equilibrium $\bar{x}$ is then

$$y_t = Ay.$$

An important question is if this linearized system provides a good local approximation of the nonlinear system for solutions that are near equilibrium. This is the case under the following condition.

DEFINITION 1.13. An equilibrium $\bar{x}$ of (1.1) is hyperbolic if $Df(\bar{x})$ has no eigenvalues with zero real part.

Thus, for a hyperbolic equilibrium, all solutions of the linearized system grow or decay exponentially in time. According to the Hartman-Grobman theorem, if $\bar{x}$ is hyperbolic, then the flows of the linearized and nonlinear system are (topologically) equivalent near the equilibrium. In particular, the stability of the nonlinear equilibrium is the same as the stability of the equilibrium of the linearized system. One has to be careful, however, in drawing conclusions about the behavior of the nonlinear system from the linearized system if $Df(\bar{x})$ has eigenvalues with zero real part. In that case the nonlinear terms may cause the growth or decay of perturbations from equilibrium, and the behavior of solutions of the nonlinear system near the equilibrium may differ qualitatively from that of the linearized system.

Non-hyperbolic equilibria are not typical for specific systems, since one does not expect the eigenvalues of a given matrix to have a real part that is exactly equal to zero. Nevertheless, non-hyperbolic equilibria arise in an essential way in bifurcation theory when an eigenvalue of a system that depends on some parameter has real part that passes through zero.

1.4. Phase space

> it may happen that small differences in the initial conditions produce very great ones in the final phenomena
>
> Henri Poincaré, 1908

Very few nonlinear systems of ODEs are explicitly solvable. Therefore, rather than looking for individual analytical solutions, we try to understand the qualitative behavior of their solutions. This global, geometrical approach was introduced by Poincaré (1880).

We may represent solutions of (1.8) by solution curves, trajectories, or orbits, $x(t)$ in phase, or state, space $\mathbb{R}^d$. These trajectories are integral curves of the vector field f, meaning that they are tangent to f at every point. The existence-uniqueness theorem implies if the vector field f is smooth, then a unique trajectory passes through each point of phase space and that trajectories cannot cross. We may visualize f as the steady velocity field of a fluid that occupies phase space and the trajectories as particle paths of the fluid.

Let $x(t; x_0)$ denote the solution of (1.8), defined on its maximal time-interval of existence $T_-(x_0) < t < T_+(x_0)$. The existence-uniqueness theorem implies that we can define a flow map, or solution map, $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi_t(x_0) = x(t; x_0), \qquad T_-(x_0) < t < T_+(x_0).$$

That is, Φ_t maps the initial data x_0 to the solution at time t. Note that $\Phi_t(x_0)$ is not defined for all $t \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ unless all solutions exist globally. In the fluid analogy, Φ_t may be interpreted as the map that takes a particle from its initial location at time 0 to its location at time t.

The flow map Φ_t of an autonomous system has the group property that

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

where $\circ$ denotes the composition of maps *i.e.* solving the ODE for time $t + s$ is equivalent to solving it for time s then for time t. We remark that the solution map of a non-autonomous IVP,

$$x_t = f(x, t), \qquad x(t_0) = x_0$$

with solution $x(t; x_0, t_0)$, is defined by

$$\Phi_{t,t_0}(x_0) = x(t; x_0, t_0).$$

The map depends on both the initial and final time, not just their difference, and satisfies

$$\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r}.$$

If $\bar{x}$ is an equilibrium solution of (1.8), with $f(\bar{x}) = 0$, then

$$\Phi_t(\bar{x}) = \bar{x},$$

which explains why equilibria are referred to as fixed points (of the flow map). We may state a precise definition of stability in terms of the flow map. There are many different, and not entirely equivalent definitions, of stability; we give only the simplest and most commonly used ones.

Definition 1.14. An equilibrium $\bar{x}$ of (1.8) is Lyapunov stable (or stable, for short) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - \bar{x}| < \delta$ then

$$|\Phi_t(x) - \bar{x}| \qquad \text{for all } t \geq 0.$$

The equilibrium is asymptotically stable if it is Lyapunov stable and there exists $\eta > 0$ such that if $|x - \bar{x}| < \eta$ then

$$\Phi_t(x) \to \bar{x} \qquad \text{as } t \to \infty.$$

Thus, stability means that solutions which start sufficiently close to the equilibrium remain arbitrarily close for all $t \geq 0$, while asymptotic stability means that in addition nearby solutions approach the equilibrium as $t \to \infty$. Lyapunov stability does not imply asymptotic stability since, for example, nearby solutions might oscillate about an equilibrium without decaying toward it. Also, it is not sufficient for asymptotic stability that all nearby solutions approach the equilibrium, because they could make large excursions before approaching the equilibrium, which would violate Lyapunov stability.

The next result implies that the solution of an IVP depends continuously on the initial data, and that the flow map of a smooth vector field is smooth. Here, 'smooth' means, for example, C^1 or C^∞.

THEOREM 1.15 (Continuous dependence on initial data). *If the vector field f in (1.8) is locally Lipschitz continuous, then the corresponding flow map $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous. Moreover, the existence times T_+ (respectively, T_-) are lower (respectively, upper) semi-continuous function of x_0. If the vector field f in (1.8) is smooth, then the corresponding flow map $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ is smooth.*

Here, the lower-semicontinuity of T_+ means that

$$T_+(x_0) \leq \liminf_{x \to x_0} T_+(x),$$

so that solutions with initial data near x_0 exist for essentially as long, or perhaps longer, than the solution with initial data x_0.

Theorem 1.15 means that solutions remain close over a finite time-interval if their initial data are sufficiently close. After long enough times, however, two solutions may diverge by an arbitrarily large amount however close their initial data.

EXAMPLE 1.16. Consider the scalar, linear ODE $x_t = x$. The solutions $x(t)$, $y(t)$ with initial data $x(0) = x_0$, $y(0) = y_0$ are given by

$$x(t) = x_0 e^t, \qquad y(t) = y_0 e^t.$$

Suppose that $[0, T]$ is any given time interval, where $T > 0$. If $|x_0 - y_0| \leq \epsilon e^{-T}$, then the solutions satisfy $|x(t) - y(t)| \leq \epsilon$ for all $0 \leq t \leq T$, so the solutions remain close on $[0, T]$, but $|x(t) - y(t)| \to \infty$ as $t \to \infty$ whenever $x_0 \neq y_0$.

Not only do the trajectories depend continuously on the initial data, but if f is Lipschitz continuous they can diverge at most exponentially quickly in time. If M is the Lipschitz constant of f and $x(t)$, $y(t)$ are two solutions of (1.8), then

$$\frac{d}{dt}|x - y| \leq M|x - y|.$$

It follows from Gronwall's inequality that if $x(0) = x_0$, $y(0) = y_0$, then

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{Mt}.$$

The local exponential divergence (or contraction) of trajectories may be different in different directions, and is measured by the Lyapunov exponents of the system. The largest such exponent is called *the* Lyapunov exponent of the system. Chaotic behavior occurs in systems with a positive Lyapunov exponent and trajectories that remain bounded; it is associated with the local exponential divergence of trajectories (essentially a linear phenomenon) followed by a global folding (typically as a result of nonlinearity).

One way to organize the study of dynamical systems is by the dimension of their phase space (following the Trolls of Discworld: one, two, three, many, and lots). In one or two dimensions, the non-intersection of trajectories strongly restricts their possible behavior: in one dimension, solutions can only increase or decrease monotonically to an equilibrium or to infinity; in two dimensions, oscillatory behavior can occur. In three or more dimensions complex behavior, including chaos, is possible.

For the most part, we will consider dynamical systems with low-dimensional phase spaces (say of dimension $d \leq 3$). The analysis of high-dimensional dynamical systems is usually very difficult, and may require (more or less well-founded) probabilistic assumptions, or continuum approximations, or some other type of approach.

Example 1.17. Consider a gas composed of N classical particles of mass m moving in three space dimensions with an interaction potential $V : \mathbb{R}^3 \to \mathbb{R}$. We denote the positions of the particles by $x = (x_1, x_2, \dots, x_N)$ and the momenta by $p = (p_1, p_2, \dots, p_N)$, where $x_i, p_i \in \mathbb{R}^3$. The Hamiltonian for this system is

$$H(x,p) = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{1\leq i\neq j\leq N} V\left(x_i - x_j\right),$$

and Hamilton's equations are

$$\frac{dx_i}{dt} = \frac{1}{m}p_i, \qquad \frac{dp_i}{dt} = -\sum_{j\neq i}\frac{\partial V}{\partial x}\left(x_i - x_j\right).$$

The phase space of this system has dimension $6N$. For a mole of gas, we have $N = N_A$ where $N_A \approx 6.02 \times 10^{23}$ is Avogadro's number, and this dimension is extremely large.

In kinetic theory, one considers equations for probability distributions of the particle locations and velocities, such as the Boltzmann equation. One can also approximate some solutions by partial differential fluid equations, such as the Navier-Stokes equations, for suitable averages.

We will mostly consider systems whose phase space is $\mathbb{R}^d$. More generally, the phase space of a dynamical system may be a manifold. We will not give the precise definition of a manifold here; roughly speaking, a d-dimensional manifold is a space that 'looks' locally like $\mathbb{R}^d$, with a d-dimensional local coordinate system about each point, but which may have a different global, topological structure. The d-dimensional sphere $\mathbb{S}^d$ is a typical example. Phase spaces that are manifolds arise, for example, if some of the state variables represent angles.

Example 1.18. The motion of an undamped pendulum of length ℓ in a gravitational field with acceleration g satisfies the pendulum equation

$$\theta_{tt} + \frac{g}{\ell}\sin\theta = 0$$

where $\theta \in \mathbb{T}$ is the angle of the pendulum to the vertical, measured in radians. Here, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denotes the circle; angles that differ by an integer multiple of 2π are equivalent. Writing the pendulum equation as a first-order system for (θ, v) where $v = \theta_t \in \mathbb{R}$ is the angular velocity, we get

$$\theta_t = v, \qquad v_t = -\frac{g}{\ell}\sin\theta$$

The phase space of this system is the cylinder $\mathbb{T} \times \mathbb{R}$. This phase space may be 'unrolled' into $\mathbb{R}^2$ with points on the θ-axis identified modulo 2π, but it is often conceptually clearer to keep the actual cylindrical structure and θ-periodicity in mind.

Example 1.19. The phase space of a rotating rigid body, such as a tumbling satellite, may be identified with the group $SO(3)$ of rotations about its center of mass from some fixed reference configuration. The three Euler angles of a rotation give one possible local coordinate system on the phase space.

Solutions of an ODE with a smooth vector field on a compact phase space without boundaries, such as $\mathbb{S}^d$, exist globally in time since they cannot escape to infinity (or hit a boundary).

1.5. Bifurcation theory

Most applications lead to equations which depend on parameters that characterize properties of the system being modeled. We write an IVP for a first-order system of ODEs for $x(t) \in \mathbb{R}^d$ depending on a vector of parameters $\mu \in \mathbb{R}^m$ as

$$\begin{aligned} x_t &= f(x;\mu), \\ x(0) &= x_0 \end{aligned} \tag{1.14}$$

where $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$.

In applications, it is important to determine a minimal set of dimensionless parameters on which the problem depends and to know what parameter regimes are relevant *e.g.* if some dimensionless parameters are very large or small.

Example 1.20. The Lorentz system (1.2) for $(x, y, z) \in \mathbb{R}^3$ depends on three parameters $(\sigma, r, \beta) \in \mathbb{R}^3$, which we assume to be positive. We typically think of fixing (σ, β) and increasing r, which in the original convection problem corresponds to fixing the fluid properties and the dimensions of the fluid layer and increasing the temperature difference across it.

If the vector field in (1.14) depends smoothly ($e.g.$ C^1 or C^∞) on the parameter μ, then so does the flow map. Explicitly, if $x(t; x_0; \mu)$ denotes the solution of (1.14), then we define the flow map Φ_t by

$$\Phi_t(x_0; \mu) = x(t; x_0; \mu).$$

Theorem 1.21 (Continuous dependence on parameters). *If the vector field $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ in (1.14) is smooth, then the corresponding flow map $\Phi_t : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is smooth.*

Bifurcation theory is concerned with changes in the qualitative behavior of the solutions of (1.14) as the parameter μ is varied. It may be difficult to carry out a full bifurcation analysis of a nonlinear dynamical system, especially when it depends on many parameters.

The simplest type of bifurcation is the bifurcation of equlbria. The equilibrium solutions of (1.14) satisfy

$$f(\bar{x}; \mu) = 0,$$

so an analysis of equilibrium bifurcations corresponds to understanding how the solutions $\bar{x}(\mu) \in \mathbb{R}^d$ of this $d \times d$ system of nonlinear, algebraic equations depend upon the parameter μ. We refer to a smooth solution $\bar{x} : I \to \mathbb{R}^d$ in a maximal domain I as a solution branch or a branch of equilibria.

There is a closely related dynamical aspect concerning how the stability of the equilibria change as the parameter μ varies. If $\bar{x}(\mu)$ is a branch of equilibrium solutions, then the linearization of the system about $\bar{x}$ is

$$x_t = A(\mu)x, \qquad A(\mu) = D_x f\left(\bar{x}(\mu); \mu\right).$$

Equilibria lose stability if some eigenvalue $\lambda(\mu)$ of A crosses from the left-half of the complex plane into the right-half plane as μ varies. By the implicit function theorem, equilibrium bifurcations are necessarily associated with a real eigenvalue passing through zero, so that A is singular at the bifurcation point.

Equilibrium bifurcations are not the only kind, and the dynamic behavior of a system may change without a change in the equilibrium solutions For example, time-periodic solutions may appear or disappear in a Hopf bifurcation, which occurs where a complex-conjugate pair of complex eigenvalues of A crosses into the right-half plane, or there may be global changes in the geometry of the trajectories in phase space, as in a homoclinic bifurcation.

1.6. Discrete dynamical systems

> Not only in research, but also in the everyday world of politics and economics, we would all be better off if more people realised that simple nonlinear systems do not necessarily possess simple dynamical properties
>
> Robert May, 1976

A (first-order, autonomous) discrete dynamical system for $x_n \in \mathbb{R}^d$ has the form

$$x_{n+1} = f(x_n) \tag{1.15}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $n \in \mathbb{Z}$ is a discrete time variable.

The orbits, or trajectories of (1.15) consist of a sequence of points $\{x_n\}$ that is obtained by iterating the map f. (They are not curves like the orbits of a continuous dynamical system.) If $f^n = f \circ f \circ \cdots \circ f$ denotes the n-fold composition of f, then

$$x_n = f^n(x_0).$$

If f is invertible, these orbits exists forward and backward in time ($n \in \mathbb{Z}$), while if f is not invertible, they exist in general only forward in time ($n \in \mathbb{N}$). An equilibrium solution $\bar{x}$ of (1.15) is a fixed point of f that satisfies

$$f(\bar{x}) = \bar{x},$$

and in that case $x_n = \bar{x}$ for all n.

A linear discrete dynamical system has the form

$$x_{n+1} = Bx_n, \tag{1.16}$$

where B is a linear transformation on $\mathbb{R}^d$. The solution is

$$x_n = B^n x_0.$$

The linear system (1.16) has the unique fixed point $\bar{x} = 0$ if $I - B$ is a nonsingular linear map. This fixed point is asymptotically stable if all eigenvalues $\lambda \in \mathbb{C}$ of B lie in the unit disc, meaning that $|\lambda| < 1$. It is unstable if B has some eigenvalue with $|\lambda| > 1$ in the exterior of the unit disc.

The linearization of (1.15) about a fixed point $\bar{x}$ is

$$x_{n+1} = Bx_n, \qquad B = Df(\bar{x}).$$

Analogously to the case of continuous systems, we can determine the stability of the fixed point from the stability of the linearized system under a suitable hyperbolicity assumption.

Definition 1.22. A fixed point $\bar{x}$ of (1.15) is hyperbolic if $Df(\bar{x})$ has no eigenvalues with absolute value equal to one.

If $\bar{x}$ is a hyperbolic fixed point of (1.15), then it is asymptotically stable if all eigenvalues of $Df(\bar{x})$ lie inside the unit disc, and unstable if some eigenvalue lies outside the unit disc.

The behavior of even one-dimensional discrete dynamical systems may be complicated. The biologist May (1976) drew attention to the fact that the logistic map,

$$x_{n+1} = \mu x_n \left(1 - x_n\right),$$

leads to a discrete dynamical system with remarkably intricate behavior, even though the corresponding continuous logistic ODE

$$x_t = \mu x(1 - x)$$

is simple to analyze completely. Another well-known illustration of the complexity of discrete dynamical systems is the fractal structure of Julia sets for complex dynamical systems (with two real dimensions) obtained by iterating rational functions $f : \mathbb{C} \to \mathbb{C}$.

Discrete dynamical systems may arise directly as models *e.g.* in population ecology, x_n might represent the population of the nth generation of species. They also arise from continuous dynamical systems.

Example 1.23. If Φ_t is the flow map of a continuous dynamical system with globally defined solutions, then the time-one map Φ_1 defines an invertible discrete dynamical system. The dimension of the discrete system is the same as the dimension of the continuous one.

Example 1.24. The time-one map of a linear system of ODEs $x_t = Ax$ is

$$B = e^A.$$

Eigenvalues of A in the left-half of the complex plane, with negative real part, map to eigenvalues of B inside the unit disc, and eigenvalues of A is the right-half-plane maps to eigenvalues of B outside the unit disc. Thus, the stability properties of the fixed point $\bar{x} = 0$ in the continuous and discrete descriptions are consistent.

Example 1.25. Consider a non-autonomous system for $x \in \mathbb{R}^d$ that depends periodically on time,

$$x_t = f(x, t), \qquad f(x, t + 1) = f(x, t).$$

We define the corresponding Poincaré map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi : x(0) \mapsto x(1).$$

Then Φ defines an autonomous discrete dynamical system of dimension d, which is one less than the dimension $d + 1$ of the original system when it is written in autonomous form. This reduction in dimension makes the dynamics of the Poincaré map easier to visualize than that the original flow, especially when $d = 2$. Moreover, by continuous dependence, trajectories of the original system remain arbitrarily close over the entire time-interval $0 \le t \le 1$ if their initial conditions are sufficient close, so replacing the full flow map by the Poincaré map does not lead to any essential loss of qualitative information.

Fixed points of the Poincaré map correspond to periodic solutions of the original system, although their minimal period need not be one; for example any solution of the original system with period $1/n$ where $n \in \mathbb{N}$ is a fixed point of the Poincaré map, as is any equilibrium solution with $f(\bar{x}, t) = 0$.

Example 1.26. Consider the forced, damped pendulum with non-dimensionalized equation

$$x_{tt} + \delta x_t + \sin x = \gamma \cos \omega t$$

where γ, δ, and ω are parameters, measuring the strength of the damping, the strength of the forcing, and the (angular) frequency of the forcing, respectively. Or a parametrically forced oscillator (such as a swing)

$$x_{tt} + (1 + \gamma \cos \omega t) \sin x = 0.$$

Here, the Poincaré map $\Phi : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is defined by

$$\Phi : (x(0), x_t(0)) \mapsto (x(T), x_t(T)), \qquad T = \frac{2\pi}{\omega}.$$

Floquet theory is concerned with such time-periodic ODEs, including the stability of their time-periodic solutions, which is equivalent to the stability of fixed points of the Poincaré map.

1.7. References

For introductory discussions of the theory of ODEs see [**1, 9**]. Detailed accounts of the mathematical theory are given in [**2, 8**]. A general introduction to nonlinear dynamical systems, with an emphasis on applications, is in [**10**]. An introduction to bifurcation theory in continuous and discrete dynamical systems is [**6**]. For a rigorous but accessible introduction to chaos in discrete dynamical systems, see [**3**]. A classic book on nonlinear dynamical systems is [**7**].

CHAPTER 2

One Dimensional Dynamical Systems

We begin by analyzing some dynamical systems with one-dimensional phase spaces, and in particular their bifurcations. All equations in this Chapter are scalar equations. We mainly consider continuous dynamical systems on the real line $\mathbb{R}$, but we also consider continuous systems on the circle $\mathbb{T}$, as well as some discrete systems.

The restriction to one-dimensional systems is not as severe as it may sound. One-dimensional systems may provide a full model of some systems, but they also arise from higher-dimensional (even infinite-dimensional) systems in circumstances where only one degree of freedom determines their dynamics. Haken used the term 'slaving' to describe how the dynamics of one set of modes may follow the dynamics of some other, smaller, set of modes, in which case the behavior of the smaller set of modes determines the essential dynamics of the full system. For example, one-dimensional equations for the bifurcation of equilibria at a simple eigenvalue may be derived by means of the Lyapunov-Schmidt reduction.

2.1. Exponential growth and decay

> I said that population, when unchecked, increased in a geometrical ratio; and subsistence for man in an arithmetical ratio.
>
> Thomas Malthus, 1798

The simplest ODE is the linear scalar equation

$$x_t = \mu x. \tag{2.1}$$

Its solution with initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{\mu t}$$

where the parameter μ is a constant. If $\mu > 0$, (2.1) describes exponential growth, and we refer to μ as the growth constant. The solution increases by a factor of e over time $T_e = 1/\mu$, and has doubling time

$$T = \frac{\log 2}{\mu}.$$

If $\mu = -\lambda < 0$, then (2.1) describes exponential decay, and we refer to $\lambda = |\mu|$ as the decay constant. The solution decreases by a factor of e over time $T_e = 1/\lambda$, and has half-life

$$T = \frac{\log 2}{\lambda}.$$

Note that μ or λ have the dimension of inverse time, as follows from equating the dimensions of the left and right hand sides of (2.1). The dimension of x is irrelevant since both sides are linear in x.

EXAMPLE 2.1. Malthus (1798) contrasted the potential exponential growth of the human population with the algebraic growth of resources needed to sustain it. An exponential growth law is too simplistic to accurately describe human populations. Nevertheless, after an initial lag period and before the limitation of nutrients, space, or other resources slows the growth, the population $x(t)$ of bacteria grown in a laboratory is well-described this law. The population doubles over the cell-division time. For example, *E. Coli* grown in glucose has a cell-division time of approximately 17 mins, corresponding to $\mu \approx 0.04\,\text{mins}^{-1}$.

EXAMPLE 2.2. Radioactive decay is well-described by (2.1) with $\mu < 0$, where $x(t)$ is the molar amount of radioactive isotope remaining after time t. For example, ^{14}C used in radioactive dating has a half-life of approximately 5730 years, corresponding to $\mu \approx -1.2 \times 10^{-4}\,\text{years}^{-1}$.

If $x_0 \geq 0$, then $x(t) \geq 0$ for all $t \in \mathbb{R}$, so this equation is consistent with modeling problems such as population growth or radioactive decay where the solution should remain non-negative. We assume also that the population or number of radioactive atoms are sufficiently large that we can describe them by a continuous variable.

The phase line of (2.1) consists of a globally asymptotically stable equilibrium $x = 0$ if $\mu < 0$, and an unstable equilibrium $x = 0$ if $\mu > 0$. If $\mu = 0$, then every point on the phase line is an equilibrium.

2.2. The logistic equation

The simplest model of population growth of a biological species that takes account of the effect of limited resources is the logistic equation

$$x_t = \mu x \left(1 - \frac{x}{K}\right). \tag{2.2}$$

Here, $x(t)$ is the population at time t, the constant $K > 0$ is called the carrying capacity of the system, and $\mu > 0$ is the maximum growth rate, which occurs at populations that are much smaller than the carrying capacity. For $0 < x < K$, the population increases, while for $x > K$, the population decreases.

We can remove the parameters μ, K by introducing dimensionless variables

$$\tilde{t} = \mu t, \qquad \tilde{x}(\tilde{t}) = \frac{x(t)}{K}.$$

Since $\mu > 0$, this transformation preserves the time-direction. The non-dimensionalized equation is $\tilde{x}_{\tilde{t}} = \tilde{x}\,(1 - \tilde{x})$ or, on dropping the tildes,

$$x_t = x(1 - x).$$

The solution of this ODE with initial condition $x(0) = x_0$ is

$$x(t) = \frac{x_0 e^t}{1 - x_0 + x_0 e^t}.$$

The phase line consists of an unstable equilibrium at $x = 0$ and a stable equilibrium at $x = 1$. For any initial data with $x_0 > 0$, the solution satisfies $x(t) \to 1$ as $t \to \infty$, meaning that the population approaches the carrying capacity.

2.3. The phase line

Consider a scalar ODE

$$x_t = f(x) \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

To sketch the phase line of this system one just has to examine the sign of f. Points where $f(x) = 0$ are equilibria. In intervals where $f(x) > 0$, solutions are increasing, and trajectories move to the right. In intervals where $f(x) < 0$, solutions are decreasing and trajectories move to the left. This gives a complete picture of the dynamics, consisting monotonically increasing or decreasing trajectories that approach equilibria, or go off to infinity.

The linearization of (2.3) at an equilibrium $\bar{x}$ is

$$x_t = ax, \qquad a = f'(\bar{x}).$$

The equilibrium is asymptotically unstable if $f'(\bar{x}) < 0$ and unstable if $f'(\bar{x}) > 0$. If $f'(\bar{x}) = 0$, meaning that the fixed point is not hyperbolic, there is no immediate conclusion about the stability of $\bar{x}$, although one can determine its stability by looking at the behavior of the sign of f near the equilibrium.

Example 2.3. The ODE

$$x_t = x^2$$

with $f(x) = x^2$ has the unique equilibrium $x = 0$, but $f'(0) = 0$. Solutions with $x(0) < 0$ approach the equilibrium as $t \to \infty$, while solutions with $x(0) > 0$ leave it (and go off to ∞ in finite time). Such a equilibrium with one-sided stability is sometimes said to be semi-stable.

Example 2.4. For the ODE $x_t = -x^3$, the equilibrium $x = 0$ is asymptotically stable, while for $x_t = x^3$ it is unstable, even though $f'(0) = 0$ in both cases. Note, however, that perturbations from the equilibrium grow or decay algebraically in time, not exponentially as in the case of a hyperbolic equilibrium.

2.4. Bifurcation theory

> And thirdly, the code is more what you'd call "guidelines" than actual rules.
>
> Barbossa

Like the pirate code, the notion of a bifurcation is more of a guideline than an actual rule. In general, it refers to a qualitative change in the behavior of a dynamical system as some parameter on which the system depends varies continuously.

Consider a scalar ODE

$$x_t = f(x; \mu) \tag{2.4}$$

depending on a single parameter $\mu \in \mathbb{R}$ where f is a smooth function. The qualitative dynamical behavior of a one-dimensional continuous dynamical system is determined by its equilibria and their stability, so all bifurcations are associated with bifurcations of equilibria. One possible definition (which does not refer directly to the stability of the equilibria) is as follows.

DEFINITION 2.5. A point (x_0, μ_0) is a bifurcation point of equilibria for (2.4) if the number of solutions of the equation $f(x;\mu) = 0$ for x in every neighborhood of (x_0, μ_0) is not a constant independent of μ.

The three most important one-dimensional equilibrium bifurcations are described locally by the following ODEs:

$$
\begin{aligned}
x_t &= \mu - x^2, && \text{saddle-node} \\
x_t &= \mu x - x^2, && \text{transcritical} \\
x_t &= \mu x - x^3, && \text{pitchfork.}
\end{aligned} \tag{2.5}
$$

We will study each of these in more detail below.

2.5. Saddle-node bifurcation

Consider the ODE

$$x_t = \mu + x^2. \tag{2.6}$$

Equations $x_t = \pm\mu \pm x^2$ with other choices of signs can be transformed into (2.6) by a suitable change in the signs of x and μ, although the transformation $\mu \mapsto -\mu$ changes increasing μ to decreasing μ.

The ODE (2.6) has two equilibria

$$x = \pm\sqrt{-\mu}$$

if $\mu < 0$, one equilibrium $x = 0$ if $\mu = 0$, and no equilibria if $\mu > 0$. For the function $f(x;\mu) = \mu + x^2$, we have

$$\frac{\partial f}{\partial x}(\pm\sqrt{-\mu};\mu) = \pm 2\sqrt{-\mu}.$$

Thus, if $\mu < 0$, the equilibrium $\sqrt{-\mu}$ is unstable and the equilibrium $-\sqrt{-\mu}$ is stable. If $\mu = 0$, then the ODE is $x_t = x^2$, and $x = 0$ is a non-hyperbolic, semi-stable equilibrium.

This bifurcation is called a saddle-node bifurcation. In it, a pair of hyperbolic equilibria, one stable and one unstable, coalesce at the bifurcation point, annihilate each other and disappear.[1] We refer to this bifurcation as a subcritical saddle-node bifurcation, since the equilibria exist for values of μ above the bifurcation value 0. With the opposite sign $x_t = \mu - x^2$, the equilibria appear at the bifurcation point $(x,\mu) = (0,0)$ as μ increases through zero, and we get a supercritical saddle-node bifurcation. A saddle-node bifurcation is the generic way in which there is a change in the number of equilibrium solutions of a dynamical system.

The name "saddle-node" comes from the corresponding two-dimensional bifurcation in the phase plane, in which a saddle point and a node coalesce and disappear, but the other dimension plays no essential role in that case and this bifurcation is one-dimensional in nature.

[1]If we were to allow complex equilibria, the equilibria would remain but become imaginary.

2.6. Transcritical bifurcation

Consider the ODE

$$x_t = \mu x - x^2.$$

This has two equilibria at $x = 0$ and $x = \mu$. For $f(x;\mu) = \mu x - x^2$, we have

$$\frac{\partial f}{\partial x}(x;\mu) = \mu - 2x, \qquad \frac{\partial f}{\partial x}(0;\mu) = \mu, \qquad \frac{\partial f}{\partial x}(\mu;\mu) = -\mu.$$

Thus, the equilibrium $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$, while the equilibrium $x = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$. Note that although $x = 0$ is asymptotically stable for $\mu < 0$, it is not globally stable: it is unstable to negative perturbations of magnitude greater than μ, which can be small near the bifurcation point.

This transcritical bifurcation arises in systems where there is some basic "trivial" solution branch, corresponding here to $x = 0$, that exists for all values of the parameter μ. (This differs from the case of a saddle-node bifurcation, where the solution branches exists locally on only one side of the bifurcation point.). There is a second solution branch $x = \mu$ that crosses the first one at the bifurcation point $(x, \mu) = (0, 0)$. When the branches cross one solution goes from stable to unstable while the other goes from stable to unstable. This phenomenon is referred to as an "exchange of stability."

2.7. Pitchfork bifurcation

Consider the ODE

$$x_t = \mu x - x^3.$$

Note that this ODE is invariant under the reflectional symmetry $x \mapsto -x$. It often describes systems with this kind of symmetry *e.g.* systems where there is no distinction between left and right.

The system has one globally asymptotically stable equilibrium $x = 0$ if $\mu \leq 0$, and three equilibria $x = 0$, $x = \pm\sqrt{\mu}$ if μ is positive. The equilibria $\pm\sqrt{\mu}$ are stable and the equilibrium $x = 0$ is unstable for $\mu > 0$. Thus the stable equilibrium 0 looses stability at the bifurcation point, and two new stable equilibria appear. The resulting pitchfork-shape bifurcation diagram gives this bifurcation its name.

This pitchfork bifurcation, in which a stable solution branch bifurcates into two new stable branches as the parameter μ is increased, is called a supercritical bifurcation. Because the ODE is symmetric under $x \mapsto -x$, we cannot normalize all the signs in the ODE without changing the sign of t, which reverses the stability of equilibria.

Up to changes in the signs of x and μ, the other distinct possibility is the subcritical pitchfork bifurcation, described by

$$x_t = \mu x + x^3.$$

In this case, we have three equilibria $x = 0$ (stable), $x = \pm\sqrt{-\mu}$ (unstable) for $\mu < 0$, and one unstable equilibrium $x = 0$ for $\mu > 0$.

A supercritical pitchfork bifurcation leads to a "soft" loss of stability, in which the system can go to nearby stable equilibria $x = \pm\sqrt{\mu}$ when the equilibrium $x = 0$ loses stability as μ passes through zero. On the other hand, a subcritical pitchfork bifurcation leads to a "hard" lose of stability, in which there are no nearby equilibria and the system goes to some far-off dynamics (or perhaps to infinity) when the equilibrium $x = 0$ loses stability.

EXAMPLE 2.6. The ODE

$$x_t = \mu x + x^3 - x^5$$

has a subcritical pitchfork bifurcation at $(x, \mu) = (0, 0)$. When the solution $x = 0$ loses stability as μ passes through zero, the system can jump to one of the distant stable equilibria with

$$x^2 = \frac{1}{2}\left(1 + \sqrt{1 + 4\mu}\right),$$

corresponding to $x = \pm 1$ at $\mu = 0$.

2.8. The implicit function theorem

The above bifurcation equations for equilibria arise as normal forms from more general bifurcation equations, and they may be derived by a suitable Taylor expansion.

Consider equilibrium solutions of (2.4) that satisfy

$$f(x; \mu) = 0 \tag{2.7}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose that x_0 is an equilibrium solution at μ_0, meaning that

$$f(x_0; \mu_0) = 0.$$

Let us look for equilibria that are close to x_0 when μ is close to μ_0. Writing

$$x = x_0 + x_1 + \dots, \qquad \mu = \mu_0 + \mu_1$$

where x_1, μ_1 are small, and Taylor expanding (2.7) up to linear terms, we get that

$$\frac{\partial f}{\partial x}(x_0; \mu_0)x_1 + \frac{\partial f}{\partial \mu}(x_0; \mu_0)\mu_1 + \dots = 0$$

where the dots denote higher-order terms (*e.g.* quadratic terms). Hence, if

$$\frac{\partial f}{\partial x}(x_0; \mu_0) \neq 0$$

we expect to be able to solve (2.7) uniquely for x when (x, μ) is sufficiently close to (x_0, μ_0), with

$$x_1 = c\mu_1 + \dots, \qquad c = -\left[\frac{\partial f / \partial \mu(x_0; \mu_0)}{\partial f / \partial x(x_0; \mu_0)}\right]. \tag{2.8}$$

This is in fact true, as stated in the following fundamental result, which is the scalar version of the implicit function theorem.

THEOREM 2.7. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function and*

$$f(x_0; \mu_0) = 0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) \neq 0.$$

Then there exist $\delta, \epsilon > 0$ and a C^1 function

$$\bar{x} : (\mu_0 - \epsilon, \mu_0 + \epsilon) \to \mathbb{R}$$

such that $x = \bar{x}(\mu)$ is the unique solution of

$$f(x; \mu) = 0$$

with $|x - x_0| < \delta$ and $|\mu - \mu_0| < \epsilon$.

By differentiating this equation

$$f(\bar{x}(\mu);\mu) = 0$$

with respect to μ, setting $\mu = \mu_0$, and solving for $d\bar{x}/d\mu$, we get that

$$\frac{d\bar{x}}{d\mu}(\mu_0) = -\left.\frac{\partial f/\partial \mu}{\partial f/\partial x}\right|_{x=x_0,\mu=\mu_0}$$

in agreement with (2.8).

For the purposes of bifurcation theory, the most important conclusion from the implicit function theorem is the following:

Corollary 2.8. *If $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a C^1-function, then a necessary condition for a solution (x_0, μ_0) of (2.7) to be a bifurcation point of equilibria is that*

$$\frac{\partial f}{\partial x}(x_0;\mu_0) = 0. \tag{2.9}$$

Another way to state this result is that hyperbolic equilibria are stable under small variations of the system, and a local bifurcation of equilibria can occur only at a non-hyperbolic equilibrium.

While an equilibrium bifurcation is typical at points where (2.9) holds, there are in exceptional, degenerate cases in which no bifurcation occurs. Thus, on its own, (2.9) is a necessary but not sufficient condition for the bifurcation of equilibria.

Example 2.9. The ODE

$$x_t = \mu - x^3$$

with $f(x;\mu) = \mu - x^3$ has a unique branch $x = (\mu)^{1/3}$ of globally stable equilibria. No bifurcation of equilibria occurs at $(0, 0)$ even though

$$\frac{\partial f}{\partial x}(0;0) = 0.$$

Note, however, that the equilibrium branch is not a C^1-function of μ at $\mu = 0$.

Example 2.10. The ODE

$$x_t = (\mu - x)^2$$

with $f(x;\mu) = (x - \mu)^2$ has a unique branch $x = \mu$ of non-hyperbolic equilibria, all of which are semi-stable. There are no equilibrium bifurcations, but

$$\frac{\partial f}{\partial x}(\mu;\mu) = 0$$

for all values of μ.

There is a close connection between the loss of stability of equilibria of (2.4) and their bifurcation. If $x = \bar{x}(\mu)$ is a branch of equilibria, then the equilibria are stable if

$$\frac{\partial f}{\partial x}(\bar{x}(\mu);\mu) < 0$$

and unstable if

$$\frac{\partial f}{\partial x}(\bar{x}(\mu);\mu) > 0.$$

It follows that if the equilibria lose stability at $\mu = \mu_0$, then $\partial f/\partial x(\bar{x}(\mu);\mu)$ changes sign at $\mu = \mu_0$ so (2.9) holds at that point. Thus, the loss of stability of a branch of equilibria due to the passage of an eigenvalue of the linearized system through zero is typically associated with the appearance or disappearance of other equilibria.[2]

When (2.9) holds, we have to look at the higher-order terms in the Taylor expansion of $f(x;\mu)$ to determine what type of bifurcation (if any) actually occurs. We can always transfer a bifurcation point at (x_0,μ_0) to $(0,0)$ by the change of variables $x \mapsto x - x_0$, $\mu \mapsto \mu - \mu_0$. Moreover, if $x = \bar{x}(\mu)$ is a solution branch, then $x \mapsto x - \bar{x}(\mu)$ maps the branch to $x = 0$.

Let us illustrate the idea with the simplest example of a saddle-node bifurcation. Suppose that

$$f(0,0) = 0, \qquad \frac{\partial f}{\partial x}(0,0) = 0$$

so that $(0,0)$ is a possible bifurcation point. Further suppose that

$$\frac{\partial f}{\partial \mu}(0;0) = a \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(0;0) = b \neq 0.$$

Then Taylor expanding $f(x;\mu)$ up to the leading-order nonzero terms in x, μ we get that

$$f(x;\mu) = a\mu + \frac{1}{2}bx^2 + \dots$$

Thus, neglecting the higher-order terms,[3] we may approximate the ODE (2.4) near the origin by

$$x_t = a\mu + \frac{1}{2}bx^2.$$

By rescaling x and μ, we may put this ODE in the standard normal form for a saddle-node bifurcation. The signs of a and b determine whether the bifurcation is subcritical or supercritical and which branches are stable or unstable. For example, if $a, b > 0$, we get the same bifurcation diagram and local dynamics as for (2.6).

As in the case of the implicit function theorem, this formal argument does not provide a rigorous proof that a saddle-node bifurcation occurs, and one has to justify the neglect of the higher-order terms. In particular, it may not be obvious which terms can be safely neglected and which terms must be retained. We will not give any further details here, but simply summarize the resulting conclusions in the following theorem.

THEOREM 2.11. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function and (x_0,μ_0) satisfy the necessary bifurcation conditions*

$$f(x_0,\mu_0) = 0, \qquad \frac{\partial f}{\partial x}(x_0;\mu_0) = 0.$$

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0;\mu_0) \neq 0$$

  then a saddle-node bifurcation occurs at (x_0,μ_0)

[2] In higher-dimensional systems, an equilibrium may lose stability by the passage of a complex conjugate pair of eigenvalues across the real axis. This does not lead to an equilibrium bifurcation, since the eigenvalues are always nonzero. Instead, as we will discuss later on, it is typically associated with the appearance or disappearance of periodic solutions in a Hopf bifurcation.

[3] For example, μx is small compared with μ and x^3 is small compared with x^2 since x is small, and μ^2 is small compared with μ since μ is small.

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0;\mu_0) \neq 0$$

then a transcritical bifurcation occurs at (x_0, μ_0).

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x^2}(x_0;\mu_0) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^3 f}{\partial x^3}(x_0;\mu_0) \neq 0$$

then a pitchfork bifurcation occurs at (x_0, μ_0)

The conditions in the theorem are rather natural; they state that the leading nonzero terms in the Taylor expansion of f agree with the terms in the corresponding normal form. Note that a saddle-node bifurcation is generic, in the sense that other derivatives of f have to vanish at the bifurcation point if a saddle-node bifurcation is not to occur.

In each case of Theorem 2.11, one can find local coordinates near (x_0, μ_0) that put the equation $f(x;\mu) = 0$ in the normal form for the corresponding bifurcation in a sufficiently small neighborhood of the bifurcation point. In particular, the bifurcation diagrams look locally like the ones considered above. The signs of the nonzero terms determine the stability of the various branches and whether or not the bifurcation is subcritical or supercritical.

Example 2.12. Bifurcation points for the ODE

$$x_t = \mu x - e^x$$

must satisfy

$$\mu x - e^x = 0, \qquad \mu - e^x = 0,$$

which implies that $(x, \mu) = (1, e)$. This can also be seen by plotting the graphs of $y = \mu x$ and $y = e^x$: the line $y = \mu x$ is tangent to the curve $y = e^x$ at $(x, y) = (1, e)$ when $\mu = e$. Writing

$$x = 1 + x_1 + \dots, \qquad \mu = e + \mu_1 + \dots$$

we find that the Taylor approximation of the ODE near the bifurcation point is

$$x_{1t} = \mu_1 - \frac{e}{2}x_1^2 + \dots.$$

Thus, there a supercritical saddle node bifurcation at $(x, \mu) = (1, e)$. For $\mu > e$, the equilibrium solutions are given by

$$x = 1 \pm \sqrt{\frac{2}{e}(\mu - e)} + \dots.$$

The solution with $x > 1$ is stable, while the solution with $x < 1$ is unstable.

Example 2.13. The ODE

$$x_t = \mu^2 + \mu x - x^3 \tag{2.10}$$

with $f(x;\mu) = \mu^2 + \mu x - x^3$ has a supercritical pitchfork bifurcation at $(x, \mu) = (0, 0)$, since it satisfies the conditions of the theorem. The quadratic term μ^2 does not affect the type of bifurcation. We will return to this equation in Example 2.14 below.

2.9. Buckling of a rod

Consider two rigid rods of length L connected by a torsional spring with spring constant k and subject to a compressive force of strength λ. If x is the angle of the rods to the horizontal, then the potential energy of the system is

$$V(x) = \frac{1}{2}kx^2 + 2\lambda L\left(\cos x - 1\right).$$

Here $kx^2/2$ is the energy required to compress the spring by an angle $2x$ and $2\lambda L\left(1 - \cos x\right)$ is the work done on the system by the external force. Equilibrium solutions satisfy $V'(x) = 0$ or

$$x - \mu \sin x = 0, \qquad \mu = \frac{2\lambda L}{k}$$

where μ is a dimensionless force parameter.

The equation has the trivial, unbuckled, solution branch $x = 0$. Writing

$$f(x;\mu) = -V'(x) = \mu \sin x - x,$$

the necessary condition for an equilibrium bifurcation to occur on this branch is

$$\frac{\partial f}{\partial x}(0,\mu) = \mu - 1 = 0$$

which occurs at $\mu = 1$. The Taylor expansion of $f(x;\mu)$ about $(0,1)$ is

$$f(x;\mu) = (\mu - 1)x - \frac{1}{6}x^3 + \dots$$

Thus, there is a supercritical pitchfork bifurcation at $(x,\mu) = (0,1)$. The bifurcating equilbria near this point are given for $0 < \mu - 1 \ll 1$ by

$$x = \sqrt{6(\mu - 1)} + \dots.$$

This behavior can also be seen by sketching the graphs of $y = x$ and $y = \mu \sin x$. Note that the potential energy V goes from a single well to a double well as μ passes through 1.

This one-dimensional equation provides a simple model for the buckling of an elastic beam, one of the first bifurcation problems which was originally studied by Euler (1757).

2.10. Imperfect bifurcations

According to Theorem 2.11, a saddle-node bifurcation is the generic bifurcation of equilibria for a one-dimensional system, and additional conditions are required at a bifurcation point to obtain a transcritical or pitchfork bifurcation. As a result, these latter bifurcations are not structurally stable and they can be destroyed by arbitrarily small perturbations that break the conditions under which they occur

First, let us consider a perturbed, or imperfect, pitchfork bifurcation that is described by

$$x_t = \lambda + \mu x - x^3 \tag{2.11}$$

where $(\lambda,\mu) \in \mathbb{R}^2$ are real parameters. Note that if $\lambda = 0$, this system has the reflectional symmetry $x \mapsto -x$ and a pirchfork bifurcation, but this symmetry is broken when $\lambda \neq 0$.

The cubic polynomial $p(x) = \lambda + \mu x - x^3$ has repeated roots if

$$\lambda + \mu x - x^3 = 0, \qquad \mu - 3x^2 = 0$$

which occurs if $\mu = 3x^2$ and $\lambda = -2x^3$ or

$$4\mu^3 = 27\lambda^2.$$

As can be seen by sketching the graph of p, there are three real roots if $\mu > 0$ and

$$27\lambda^2 < 4\mu^3,$$

and one real root if $27\lambda^2 > 4\mu^3$. The surface of the roots as a function of (λ, μ) forms a cusp catastrophe.

If $\lambda \neq 0$, the pitchfork bifurcation is perturbed to a stable branch that exists for all values of μ without any bifurcations and a supercritical saddle-node bifurcation in which the remaining stable and unstable branches appear.

EXAMPLE 2.14. The ODE (2.10) corresponds to (2.11) with $\lambda = \mu^2$. As this parabola passes through the origin in the (λ, μ)-plane, we get a supercritical pitchfork bifurcation at $(x, \mu) = (0, 0)$. We then get a further saddle-node bifurcation at $(x, \mu) = (-2/9, 4/27)$ when the parabola $\lambda = \mu^2$ crosses the curve $4\mu^3 = 27\lambda^2$.

Second, consider an imperfect transcritical bifurcation described by

$$x_t = \lambda + \mu x - x^2 \tag{2.12}$$

where $(\lambda, \mu) \in \mathbb{R}^2$ are real parameters. Note that if $\lambda = 0$, this system has the equilibrium solution $x = 0$, but if $\lambda < 0$ there is no solution branch that is defined for all values of μ.

The equilibrium solutions of (2.12) are

$$x = \frac{1}{2}\left(\mu \pm \sqrt{\mu^2 + 4\lambda}\right),$$

which are real provided that $\mu^2 + 4\lambda \geq 0$. If $\lambda < 0$, the transcritical bifurcation for $\lambda = 0$ is perturbed into two saddle-node bifurcations at $\mu = \pm 2\sqrt{-\lambda}$; while if $\lambda > 0$, we get two non-intersecting solution branches, one stable and one unstable, and no bifurcations occur as μ is varied.

2.11. Dynamical systems on the circle

Problems in which the dependent variable $x(t) \in \mathbb{T}$ is an angle, such as the phase of an oscillation, lead to dynamical systems on the circle.

As an example, consider a forced, highly damped pendulum. The equation of motion of a linearly damped pendulum of mass m and length ℓ with angle $x(t)$ to the vertical acted on by a constant angular force F is

$$m\ell x_{tt} + \delta x_t + mg \sin x = F$$

where δ is a positive damping coefficient and g is the acceleration due to gravity.

The damping coefficient δ has the dimension of Force $\times$ Time. For motions in which the damping and gravitational forces are important, an appropriate time scale is therefore δ/mg, and we introduce a dimensionless time variable

$$\tilde{t} = \frac{mg}{\delta} t, \qquad \frac{d}{dt} = \frac{mg}{\delta}\frac{d}{d\tilde{t}}.$$

The angle x is already dimensionless, so we get the non-dimensionalized equation

$$\epsilon x_{\tilde{t}\tilde{t}} + x_{\tilde{t}} + \sin x = \mu$$

where the dimensionless parameters ϵ, μ are given by

$$\epsilon = \frac{m^2 g\ell}{\delta^2}, \qquad \mu = \frac{F}{mg}.$$

For highly damped motions, we neglect the term $\epsilon x_{\tilde{t}\tilde{t}}$ and set $\epsilon = 0$. Note that one has to be careful with such an approximation: the higher-order derivative is a singular perturbation, and it may have a significant effect even though its coefficient is small. For example, the second order ODE with $\epsilon > 0$ requires two initial conditions, whereas the first order ODE for $\epsilon = 0$ requires only one. Thus for the reduced equation with $\epsilon = 0$ we can specify the initial location of the pendulum, but we cannot specify its initial velocity, which is determined by the ODE. We will return to such questions in more detail later on, but for now we simply set $\epsilon = 0$.

Dropping the tilde on $\tilde{t}$ we then get the ODE

$$x_t = \mu - \sin x,$$

where μ is a nondimensionalized force parameter. If $\mu = 0$, this system has two equilibria: a stable one at $x = 0$ corresponding to the pendulum hanging down, and an unstable one at $\mu = \pi$ corresponding to the pendulum balanced exactly above its fulcrum. As μ increases, these equilibria move toward each other (the stable equilibrium is 'lifted up' by the external force), and when $\mu = 1$ they coalesce and disappear in a saddle-node bifurcation at $(x, \mu) = (\pi/2, 1)$. For $\mu > 1$, there are no equilibria. The external force is sufficiently strong to overcome the damping and the pendulum rotates continuously about its fulcrum.

2.12. Discrete dynamical systems

A one-dimensional discrete dynamical system

$$x_{n+1} = f(x_n) \tag{2.13}$$

is given by iterating a map f, which we assume is smooth. Its equilibria are fixed points $\bar{x}$ of f such that

$$\bar{x} = f(\bar{x}).$$

The orbits, or trajectories, of (2.13) consist of a sequence of points x_n rather than a curve $x(t)$ as in the case of an ODE. As a result, there are no topological restrictions on trajectories of a discrete dynamical systems, and unlike the continuous case there is no simple, general way to determine their phase portrait. In fact, their behavior may be extremely complex, as the logistic map discussed in Section 2.15 illustrates.

There is a useful graphical way to sketch trajectories of (2.13): Draw the graphs $y = f(x)$, $y = x$ and iterate points vertically to $y = f(x)$, which updates the state, and horizontally to $y = x$, which updates x-value.

The simplest discrete dynamical system is the linear scalar equation

$$x_{n+1} = \mu x_n. \tag{2.14}$$

The solution is

$$x_n = \mu^n x_0.$$

If $\mu \neq 1$, the origin $x = 0$ is the unique fixed point of the system. If $|\mu| < 1$, this fixed point is globally asymptotically stable, while if $|\mu| > 1$ it is unstable. Note that if $\mu > 0$, successive iterates approach or leave the origin monotonically, while if $\mu < 0$, they alternate on either side of the origin. If $\mu = 1$, then every point is a fixed point of (2.14), while if $\mu = -1$, then every point has period two. The map is invertible if $\mu \neq 0$ when orbits are defined backward and forward in time. If $\mu = 0$, every point is mapped to the origin after one iteration, and orbits are not defined backward in time.

EXAMPLE 2.15. The exponential growth of a population of bacteria that doubles every generation is described by (2.14) with $\mu = 2$ where x_n denotes the population of the nth generation.

The linearization of (2.13) about a fixed point $\bar{x}$ is

$$x_{n+1} = ax_n, \qquad a = f'(\bar{x})$$

where the prime denotes an x-derivative. We say that the fixed point is hyperbolic if $|f'(\bar{x})| \neq 1$, and in that case it is stable if

$$|f'(\bar{x})| < 1$$

and unstable if

$$|f'(\bar{x})| > 1.$$

As for continuous dynamical systems, the stability of non-hyperbolic equilibria (with $f'(\bar{x}) = \pm 1$) cannot be determined solely from their linearization.

After fixed points, the next simplest type of solution of (2.13) are periodic solutions. A state x_1 or x_2 has period two if the system has an orbit of the form $\{x_1, x_2\}$ where

$$x_2 = f(x_1), \qquad x_1 = f(x_2).$$

The system oscillates back and forth between the two states x_1, x_2.

We can express periodic orbits as fixed points of a suitable map. We write the composition of f with itself as $f^2 = f \circ f$, meaning that

$$f^2(x) = f\left(f(x)\right).$$

Note that this is not the same as the square of f — for example $\sin^2(x) = \sin(\sin x)$ not $(\sin x)^2$ — but our use of the notation should be clear from the context. If $\{x_1, x_2\}$ is a period-two orbit of f, then x_1, x_2 are fixed points of f^2 since

$$f^2(x_1) = f\left(f(x_1)\right) = f(x_2) = x_1.$$

Conversely, if x_1 is a fixed point of f^2 and $x_2 = f(x_1)$, then $\{x_1, x_2\}$ is a period-two orbit of (2.13).

More generally, for any $N \in \mathbb{N}$, a period-N orbit of (2.13) consists of points $\{x_1, x_2, x_3, \dots, x_N\}$ such that

$$x_2 = f(x_1), \qquad x_3 = f(x_2), \qquad \dots, \qquad x_1 = f(x_N).$$

In that case, each x_i is an N-periodic solution of (2.13) and is a fixed point of f^N, the N-fold composition of f with itself. If a point has period N, then it also has period equal to every positive integer multiple of N. For example, a fixed point has period equal to every positive integer, while a point with period two has period equal to every even positive integer.If x_1 is a periodic solution of (2.13), the minimal period of x_1 is the smallest positive integer N such that $f^N(x_1) = x_1$. Thus, for example, the fixed points of f^4 include all fixed points of f and all two-periodic points as well as all points whose minimal period is four.

2.13. Bifurcations of fixed points

Next, we consider some bifurcations of a one-dimensional discrete dynamical system

$$x_{n+1} = f(x_n; \mu) \tag{2.15}$$

depending on a parameter $\mu \in \mathbb{R}$. As usual, we assume that f is a smooth function. In addition to local bifurcations of fixed points that are entirely analogous to bifurcations of equilibria in continuous dynamical systems, these systems possess a period-doubling, or flip, bifurcation that has no continuous analog. They also possess other, more complex, bifurcations. First, we consider bifurcations of fixed points.

If x_0 is a fixed point of (2.15) at $\mu = \mu_0$, then by the implicit function theorem the fixed-point equation

$$f(x; \mu) - x = 0$$

is uniquely solvable for x close to x_0 and μ close to μ_0 provided that

$$\frac{\partial f}{\partial x}(x_0; \mu_0) - 1 \neq 0.$$

Thus, necessary conditions for $(x_0; \mu_0)$ to be a bifurcation point of fixed points are that

$$f(x_0; \mu_0) = x_0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) = 1.$$

The following typical bifurcations at $(x, \mu) = (0, 0)$ are entirely analogous to the ones in (2.5):

$$\begin{aligned} x_{n+1} &= \mu + x_n - x_n^2, && \text{saddle-node} \\ x_{n+1} &= (1+\mu)x_n - x_n^2, && \text{transcritical} \\ x_{n+1} &= (1+\mu)x_n - x_n^3, && \text{pitchfork.} \end{aligned}$$

For completeness, we state the theorem for fixed-point bifurcations, in which the equation $f(x; \mu) - x = 0$ replaces the equation $f(x; \mu) = 0$ for equilibria.

THEOREM 2.16. *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function and (x_0, μ_0) satisfies the necessary condition for the bifurcation of fixed points:*

$$f(x_0, \mu_0) = x_0, \qquad \frac{\partial f}{\partial x}(x_0; \mu_0) = 1.$$

- *If*
$$\frac{\partial f}{\partial \mu}(x_0; \mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0; \mu_0) \neq 0$$
then a saddle-node bifurcation occurs at (x_0, μ_0)
- *If*
$$\frac{\partial f}{\partial \mu}(x_0; \mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x \partial \mu}(x_0; \mu_0) \neq 0, \qquad \frac{\partial^2 f}{\partial^2 x}(x_0; \mu_0) \neq 0$$
then a transcritical bifurcation occurs at (x_0, μ_0).

- *If*

$$\frac{\partial f}{\partial \mu}(x_0;\mu_0) = 0, \qquad \frac{\partial^2 f}{\partial x^2}(x_0;\mu_0) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(x_0;\mu_0) \neq 0, \qquad \frac{\partial^3 f}{\partial x^3}(x_0;\mu_0) \neq 0$$

then a pitchfork bifurcation occurs at (x_0, μ_0)

2.14. The period-doubling bifurcation

Suppose that (2.15) has a branch of fixed points $x = \bar{x}(\mu)$ such that

$$\bar{x}(\mu) = f\left(\bar{x}(\mu); \mu\right).$$

The fixed point can lose stability in two ways: (a) the eigenvalue $f_x\left(\bar{x}(\mu);\mu\right)$ passes through 1; (b) the eigenvalue $f_x\left(\bar{x}(\mu);\mu\right)$ passes through -1. In the first case, we typically get a bifurcation of fixed points, but in the second case the implicit function theorem implies that no such bifurcation occurs. Instead, the loss of stability is typically associated with the appearance or disappearance of period two orbits near the fixed point.

To illustrate this, we consider the following system

$$x_{n+1} = -(1+\mu)x_n + x_n^3$$

with

$$f(x;\mu) = -(1+\mu)x + x^3$$

The fixed points satisfy $x = -(1+\mu)x + x^3$, whose solutions are $x = 0$ and

$$x = \pm\sqrt{2+\mu} \tag{2.16}$$

for $\mu > -2$. We have

$$\frac{\partial f}{\partial x}(0;\mu) = -(1+\mu),$$

so the fixed point $x = 0$ is stable if $-2 < \mu < 0$ and unstable if $\mu > 0$ or $\mu < -2$.

The eigenvalue $f_x\left(0;\mu\right)$ passes through 1 at $\mu = -2$, and $x = 0$ gains stability at a supercritical pitchfork bifurcation in which the two new fixed points (2.16) appear. Note that for $\mu > -2$

$$\frac{\partial f}{\partial x}(\pm\sqrt{2+\mu};\mu) = 5 + 2\mu > 1$$

so these new fixed points are unstable.

The eigenvalue $f_x\left(0;\mu\right)$ passes through -1 at $\mu = 0$, and $x = 0$ loses stability at that point. As follows from the implicit function theorem, there is no bifurcation of fixed points: the only other branches of fixed points are (2.16), equal to $x = \pm\sqrt{2}$ at $\mu = 0$, which are far away from $x = 0$. Instead, we claim that a new orbit of period two appears at the bifurcation point.

To show this, we analyze the fixed points of the two-fold composition of f

$$\begin{aligned} f^2(x;\mu) &= -(1+\mu)\left[-(1+\mu)x + x^3\right] + \left[-(1+\mu)x + x^3\right]^3 \\ &= (1+\mu)^2 x - (1+\mu)(2+2\mu+\mu^2)x^3 + 3(1+\mu)^2x^5 - 3(1+\mu)x^7 + x^9. \end{aligned}$$

The period-doubling bifurcation for f corresponds to a pitchfork bifurcation for f^2. Near the bifurcation point $(x,\mu) = (0,0)$, we may approximate f^2 by

$$f^2(x;\mu) = (1+2\mu)x - 2x^3 + \dots.$$

The fixed points of f^2 are therefore given approximately by

$$x = \pm\sqrt{\mu},$$

and they are stable. The corresponding stable period-two orbit is $\{\sqrt{\mu}, -\sqrt{\mu}\}$.

Note that if the original equation was

$$x_{n+1} = -(1+\mu)x_n + x_n^2,$$

with a quadratically nonlinear term instead of a cubically nonlinear term, then we would still get a pitchfork bifurcation in f^2 at $(x,\mu) = (0,0)$.

2.15. The logistic map

The discrete logistic equation is

$$x_{n+1} = \mu x_n \left(1 - x_n\right), \tag{2.17}$$

which is (2.13) with

$$f(x;\mu) = \mu x(1-x).$$

We can interpret (2.17) as a model of population growth in which x_n is the population of generation n. In general, positive values of x_n may map to negative values of x_{n+1}, which would not make sense when using the logistic map as a population model. We will restrict attention to $1 \le \mu \le 4$, in which case $f(\cdot;\mu)$ maps points in $[0,1]$ into $[0,1]$. Then the population x_{n+1} is $\mu(1-x_n)$ times the population x_n of the previous generation. If $0 \le x_n < 1 - 1/\mu$, the population increases, whereas if If $1 - 1/\mu < x_n \le 1$, the population decreases. Superficially, (2.17) may appear similar to the logistic ODE (2.2), but its qualitative properties are very different. In particular, note that the quadratic logistic map is not monotone or invertible on $[0,1]$ and is typically two-to-one.

Equation (2.17) has two branches of fixed points,

$$x = 0, \qquad x = 1 - \frac{1}{\mu}.$$

We have

$$\frac{\partial f}{\partial x}(x,\mu) = \mu(1-2x)$$

so that

$$\frac{\partial f}{\partial x}(0,\mu) = \mu, \qquad \frac{\partial f}{\partial x}\left(1 - \frac{1}{\mu}, \mu\right) = 2 - \mu.$$

Thus, the fixed point $x = 0$ is unstable for $\mu > 1$. The fixed point $x = 1 - 1/\mu$ is stable for $1 < \mu < 3$ and unstable $\mu > 3$. There is a transcritical bifurcation of fixed points at $(x,\mu) = (0,1)$ where these two branches exchange stability. The fixed point $x = 1 - 1/\mu$ loses stability at $\mu = 3$ in a supercritical period doubling bifurcation as $f_x = 2 - \mu$ passes through -1.

We will not carry out a further analysis of (2.17) here. We note, however, that there is a sequence of supercritical period doubling bifurcations corresponding to the appearance of stable periodic orbits of order $2, 4, 8, \dots, 2^k, \dots$ at $\mu = \mu_k$. These bifurcation points have a finite limit

$$\mu_\infty = \lim_{k\to\infty} \mu_k \approx 3.570.$$

The bifurcation values approach μ_∞ geometrically, with

$$\lim_{k\to\infty} \left(\frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}\right) = 4.6692\dots$$

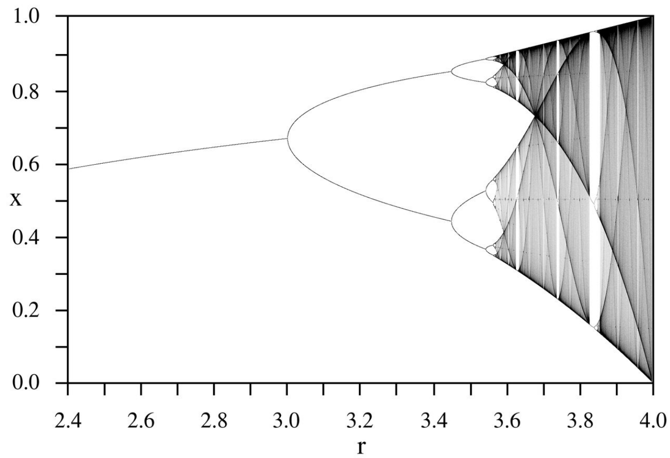

FIGURE 1. A bifurcation diagram for the logistic map where $r = \mu$. (From Wikipedia.)

where $4.6692\ldots$ is a universal Feigenbaum constant. For $\mu_\infty < \mu \leq 4$, the logistic map is chaotic, punctated by windows in which it has an asymptotically stable periodic orbit of (rather remarkably) period three.

A bifurcation diagram for (2.17) is shown in Figure 1.

2.16. References

We have studied some basic examples of equilibrium bifurcations, but have not attempted to give a general analysis, which is part of singularity or catastrophe theory. Further introductory discussions can be found in [**4, 11**]. For a systematic account, including equilibrium bifurcations in the presence of symmetry, see [**5**].

Bibliography

[1] V. I. Arnold, *Ordinary Differential Equations*, Spinger-Verlag, 1992.
[2] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger, 1984.
[3] R. Devaney and R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd Ed., Westview Press, 2003.
[4] P. Glendinning, *Stability, Instability and Chaos*, Cambridge University Press, Cambridge, 1994.
[5] M. Golubitsky and D. G. Shaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. 1, Springer-Verlag, New York, 1985.
[6] J. Hale and H. Koçak, *Dynamics and Bifurcations*, Springer-Verlag, New York, 1991.
[7] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 2nd Ed., Springer-Verlag, 1983.
[8] P. Hartman, *Ordinary Differential Equations*, 2nd Ed., Birkhauser, 1982.
[9] M. W. Hirsch, S. Smale, and R. L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, 2nd Ed., Elsevier, 2004.
[10] S. Strogatz, *Nonlinear Dynamics And Chaos*, Westview Press, 2001.
[11] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 1990.